Due Date: July 11, 2011 Monday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 302 Complex Analysis II – Homework 7 – Solutions

1	2	TOTAL
10	10	20

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. **A correct answer without proper or with too much reasoning may not get any credit.**

NAME: STUDENT NO:

Q-1) Find an infinite product expression for $\cos z$ and prove in detail all phases of your arguments.

Solution:

Let λ_k be the sequence $\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, \ldots$. Note that $\sum 1/|\lambda_k|$ diverges but $\sum 1/|\lambda_k|^2$ converges. By Weierstrass theorem, and the *Note* following it on page 219, we can write an entire function which vanishes only at these λ_k as

$$f(z) = \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{\lambda_k}\right) e^{z/\lambda_k} \right] = \prod_{k=0}^{\infty} \left(1 - \frac{4z^2}{(2k+1)^2}\right).$$

Now the arguments of Proposition 17.8 on page 221 work almost verbatim except that you should consider the figure on page 222 as shifted to the left by 1/2 first and then replace $\cos \pi z$ in the arguments with $\sin(\pi z + \pi/2)$. The arguments there now give the fact that

$$Q(z) = \frac{\prod_{k=0}^{\infty} \left(1 - \frac{4z^2}{(2k+1)^2}\right)}{\cos \pi z} = \frac{\prod_{k=0}^{\infty} \left(1 - \frac{4z^2}{(2k+1)^2}\right)}{\sin(\pi z + \pi/2)}$$

is constant. The constant is easily seen to be $Q(0) = 1$.

Alternatively you can use the identity $\cos \pi z = \dfrac{\sin 2\pi z}{2 \sin \pi z}$.

Finally replacing z by z/π, you get the infinite product expression of $\cos z$ as

$$\cos z = \prod_{k=0}^{\infty} \left(1 - \frac{4z^2}{\pi^2 (2k+1)^2}\right).$$

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Q-2) Prove that

$$\pi = 2 \prod_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{n^2}{(n-1)(n+1)}.$$

Solution:

This follows from the following simple equalities.

$$\begin{aligned} 1 &= \sin \frac{\pi}{2} \\ &= \frac{\pi}{2} \prod_{k=1}^{\infty} \left(1 - \frac{1}{4k^2}\right) \\ &= \frac{\pi}{2} \prod_{k=1}^{\infty} \left(\frac{(2k-1)(2k+1)}{(2k)^2}\right). \end{aligned}$$