Date: 28 July 2011, Thursday NAME:...
Time: 10:00-12:00
Ali Sinan Sertöz STUDENT NO:...

Math 302 Complex Analysis II – Make-Up Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Use the following at your own risk.

$$\begin{aligned}
\tan z &= \sum_{k=1}^{\infty} \frac{|B_{2k}|\, 2^{2k}(2^{2k}-1)}{(2k)!}\, z^{2k-1}, \ \ |z| < \pi/2. \\
\cot z &= \frac{1}{z} - \sum_{k=1}^{\infty} \frac{4^k\, |B_{2k}|}{(2k)!}\, z^{2k-1}, \ \ 0 < |z| < \pi. \\
\sec z &= \sum_{k=0}^{\infty} (-1)^k \frac{E_{2k}}{(2k)!}\, z^{2k}, \ \ |z| < \pi/2. \\
\operatorname{cosec} z &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(2^{2k}-2)|B_{2k}|}{(2k)!}\, z^{2k-1}, \ \ 0 < |z| < \pi.
\end{aligned}$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42},\ B_7 = 0,\ B_8 = -\frac{1}{30}.$$

$$E_0 = 1,\ E_1 = 0,\ E_2 = -1,\ E_3 = 0,\ E_4 = 5,\ E_5 = 0,\ E_6 = -61,\ E_7 = 0,\ E_8 = 1385.$$

NAME: STUDENT NO:

Q-1) Find the value of the following sum and write its exact value in the space provided. No partials!

$$\sum_{n=1}^{\infty} \frac{1}{n^6} =$$

Solution:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^6} &= -\frac{1}{2} \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^6}; z=0\right) \\
&= \frac{1}{2} \frac{4^3}{6!} |B_6| \pi^6 \\
&= \frac{\pi^6}{945}.
\end{aligned}$$

NAME: STUDENT NO:

Q-2) Evaluate the integral $\int_{27-i\infty}^{27+i\infty} \frac{14^z}{(2z+1)^6}\,dz$, where the principal branch of $\log$ is used in 14^z.

Solution:

Let $f(z) = 14^z$. Then $\int_{27-i\infty}^{27+i\infty} \frac{f(z)}{(2z+1)^6}\,dz = 2\pi i\,\mathrm{Res}\left(\frac{f(z)}{(2z+1)^6}; z = -1/2\right) = 2\pi i\,\frac{f^{(5)}(-1/2)}{5!\,2^6}$.

(For an explanation, see the first example on chapter 12.)

Since $f^{(5)}(-1/2) = (\log 14)^5\,14^{-1/2}$, the answer is $2\pi i\,\frac{(\log 14)^5\,14^{-1/2}}{5!\,2^6} \approx 0.0279i$.

NAME: STUDENT NO:

Q-3) Let D be the unit disc around the origin. Find explicitly a function $f(x, y)$ such that f is harmonic on D and $f|_{\partial D} = x^3 y$. Can we arrange it so that $f(0, 0) = 1$? Explain why or how.

Solution:

Let $u(x, y) = \operatorname{Im} z^4 = 4x^3 y - 4xy^3$. Being the imaginary part of an analytic function, $u(x, y)$ is harmonic on D.

On the unit circle we have $u|_{\partial D} = 8x^3 y - 4xy$.

Observe that xy is already harmonic everywhere.

We now have $x^3 y = \frac{1}{8} u|_{\partial D} + \frac{1}{2} xy$.

Hence we can take $f(x, y) = \frac{1}{8} u(x, y) + \frac{1}{2} xy = \frac{1}{2}(x^3 y - xy^3 + xy)$.

The general theory gives this as the unique function satisfying the requirements. Since $f(0, 0) = 0$, it cannot be changed.

NAME: STUDENT NO:

Q-4) While trying to extend the Gamma function to the whole plane we made use of the following function

$$f(z) = \frac{1}{z} - \frac{1}{(z+1)} + \frac{1}{2!(z+2)} - \cdots + \frac{(-1)^n}{n!(z+n)} + \cdots .$$

We claimed that "*$f(z)$ is an analytic function for all $z \in \mathbb{C}$ except when $z = 0, -1, -2, \ldots$.*" Prove this claim.

Solution:

Let $\sigma_n(z) = \dfrac{(-1)^n}{n!(z+n)}$. Let D be any compact region in the plane not including any of the points $z = 0, -1, -2, \ldots$. Let $\delta = \inf\{|z-k| \mid z \in D \text{ and } k = 0, -1, -2, \ldots\}$.

Since D is bounded, it is included in a disk of radius $R > 0$ around the origin. There are only finitely many integers of the form $0, -1, -2, \ldots$ inside this disk. Around each such integer there is an open disk not intersecting D since D is closed. The smallest of these finitely many positive radii is $\leq \delta$, hence $\delta > 0$.

Let $\epsilon > 0$ be given. For any $z \in D$, $|\sigma_n(z)| = \dfrac{1}{n!|z+n|} \leq \dfrac{1}{n!\delta}$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!\delta}$ converges. It now follows by Weierstrass M-test that the convergence of $\displaystyle\sum_{n=0}^{\infty} \sigma_n(z)$ to $f(z)$ is uniform.

We showed that $\displaystyle\sum_{n=0}^{\infty} \sigma_n(z)$ converges uniformly to $f(z)$ on compacta. Since each $\sigma_n(z)$ is analytic, except at $z = 0, -1, -2, \ldots$, so is $f(z)$.

NAME: STUDENT NO:

Q-5) Prove that $\displaystyle\sum_{\substack{p:\text{ prime}\\ n\geq 2}} \frac{1}{np^{nz}}$ is analytic in $\operatorname{Re} z > \frac{1}{2}$.

Show your work in detail, explain all your arguments.

Solution: Any finite sum of this expression is an entire function. It remains to show that the infinite sum converges uniformly on compacta for $\operatorname{Re} z > 1/2$. Let $z = x + iy$ and $x > 1/2$.

$$\begin{aligned}
\sum_{n=2}^{\infty} \left| \frac{1}{np^{nz}} \right| &= \sum_{n=2}^{\infty} \frac{1}{np^{nx}} \\
&\leq \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{p^{nx}} \\
&= \frac{1}{2} \left(\frac{1}{1 - 1/p^x} - 1 - \frac{1}{p^x} \right) \\
&= \frac{1}{2p^{2x}} \frac{p^x}{p^x - 1} \\
&= \frac{1}{2p^{2x}} \left(1 + \frac{1}{p^x - 1} \right) \\
&\leq \frac{1}{2p^{2x}} \left(1 + \frac{1}{2^{1/2} - 1} \right) \\
&= \frac{1}{2p^{2x}} (3.4141\ldots) \\
&< \frac{2}{p^{2x}}.
\end{aligned}$$

Let D be any compact subset of $\operatorname{Re} z > 1/2$. There is a $\delta > 0$ such that for each $z \in D$, $x \geq 1/2 + \delta$. Then $\frac{2}{p^{2x}} \leq \frac{2}{p^{1+2\delta}}$. Since $\displaystyle\sum_{p:\text{ prime}} \frac{2}{p^{1+2\delta}}$ converges, it follows by Weierstrass M-test that $\displaystyle\sum_{\substack{p:\text{ prime}\\ n\geq 2}} \frac{1}{np^{nz}}$ converges uniformly. This completes the proof.