Date: 24 June 2011, Friday
Time: 13:40-15:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Midterm Exam 1 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Use the following at your own risk.

$$
\begin{aligned}
\tan z &= \sum_{k=1}^{\infty} \frac{|B_{2k}|\, 2^{2k}(2^{2k}-1)}{(2k)!}\, z^{2k-1}, \;\; |z| < \pi/2. \\
\cot z &= \frac{1}{z} - \sum_{k=1}^{\infty} \frac{4^k\, |B_{2k}|}{(2k)!}\, z^{2k-1}, \;\; 0 < |z| < \pi. \\
\sec z &= \sum_{k=0}^{\infty} \frac{E_k}{(2k)!}\, z^{2k}, \;\; |z| < \pi/2. \\
\operatorname{cosec} z &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(2^{2k}-2)|B_{2k}|}{(2k)!}\, z^{2k-1}, \;\; 0 < |z| < \pi.
\end{aligned}
$$

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Q-1)

a) Explain in detail, without proving your claims, how you calculate the sum $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ using residue theory, where $k \in \mathbb{N}^+$.

b) Using the formulas given on the cover page, write explicitly the value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$, where $k \in \mathbb{N}^+$.

Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{1}{2} \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^{2k}}; 0\right) = \frac{2^{2k-1} |B_{2k}| \pi^{2k}}{(2k)!}, \quad k \in \mathbb{N}^+.$$

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Q-2) Evaluate the integral $\displaystyle\int_{\pi-i\infty}^{\pi+i\infty} \frac{3^z}{z^{k+1}}\,dz$, where $k \in \mathbb{N}$ and the principal branch of $\log$ is used in 3^z.

Bonus (extra 10 points): Suppose we use the branch $-3\pi < \theta \leq -\pi$ for $\log$ in calculating 3^z. Does the value of the above integral change? If your answer is *no*, explain why. If your answer is *yes*, calculate the new value.

Solution:

Let $f(z) = 3^z$. Then $\displaystyle\int_{\pi-i\infty}^{\pi+i\infty} \frac{3^z}{z^{k+1}}\,dz = 2\pi i\,\mathrm{Res}\left(\frac{f(z)}{z^{2k+1}};0\right) = 2\pi i\,\frac{f^{(k)}(0)}{k!}$.

Let $-\pi < \theta_p \leq \pi$ be the principal branch of $\log$ function. Let $\theta = \theta_p + \alpha$ be another branch. In our case the second branch is given by $\alpha = -2\pi$ and for 3, the principal branch gives $\theta_p = 0$.

$3^z = \exp(z\log 3) = \exp(z\ln 3)$ for the principal branch and $3^z = \exp(z\log 3) = \exp(z[\ln 3 - 2\pi i])$ for the other branch. Then $f^{(k)}(0) = (\ln 3)^k$ for the principal branch and $f^{(k)}(0) = (\ln 3 - 2\pi i)^k$ for the other branch.

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Q-3) Given four distinct points z_1, z_2, z_3, z_4 in $\mathbb{C} \cup \{\infty\}$, let T be the unique Mobius transformation sending z_1, z_2, z_3 to $\infty, 0, 1$ in that order. We let $\langle z_1, z_2, z_3, z_4 \rangle := Tz_4$ and call it the cross-ratio of the four-tuple z_1, z_2, z_3, z_4.

a) Calculate $\langle 1, i, -i, -1 \rangle$.
b) Let S be any Mobius transformation. Prove or disprove that $\langle z_1, z_2, z_3, z_4 \rangle = \langle Sz_1, Sz_2, Sz_3, Sz_4 \rangle$ for any four-tuple of distinct points z_1, z_2, z_3, z_4 in $\mathbb{C} \cup \{\infty\}$.

Solution:

$\langle 1, i, -i, -1 \rangle = T(-1)$ where $T(z) = \dfrac{z - i}{z - 1} \cdot \dfrac{-i - 1}{-i - i}$. Then $T(-1) = 1/2$.

Let T be the unique Mobius transformation sending z_1, z_2, z_3 to $\infty, 0, 1$ in that order. Then $T \circ S^{-1}$ is the unique transformation that sends Sz_1, Sz_2, Sz_3 to $\infty, 0, 1$ in that order. By definition $\langle Sz_1, Sz_2, Sz_3, Sz_4 \rangle = T \circ S^{-1}(Sz_4) = T(z_4) = \langle z_1, z_2, z_3, z_4 \rangle$.

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Q-4) The value of an analytic function $f(z)$ at $z = \infty$ is defined to be the value of $f(1/t)$ at $t = 0$ as an element of $\mathbb{C} \cup \{\infty\}$. We can then consider a meromorphic function as a function from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$. Suppose that the Laurent expansion at the origin of such a meromorphic function is of the form

$$\frac{b_N}{z^N} + \cdots + \frac{b_1}{z} + a_0 + a_1 z + \cdots$$

where $b_N \neq 0$ and the series converges for $0 < |z| < \infty$.

Further assume that $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is one-to-one.

Prove or disprove that f is a Mobius transformation.

Solution:

Let $g(z) = b_N + b_{N-1}z + \cdots + b_1 z^N + a_0 z^{N+1} + \cdots$. Since g is analytic and $g(0) \neq 0$, using the principal branch of logarithm, we can construct an analytic function $h(z) = \exp(\frac{1}{N} \log g(z))$ such that $h(z)^N = g(z)$. Then changing coordinate from z to $w = h(z)/z$, the function f becomes $f(w) = 1/w^N$. Since f is one-to-one, N must be 1.

Now $f(z) = \dfrac{b_1 + a_0 z + a_1 z^2 + \cdots}{z}$. Since ∞ is already taken at $z = 0$, it should not be taken at $t = 0$ again where $z = 1/t$. This forces $a_j = 0$ for $j > 0$, and finally we have $f(z) = \dfrac{b_1}{z} + a_0$ which is a Mobius transformation.

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Q-5) Riemann mapping theorem states that any two simply connected, open, *proper subsets* of $\mathbb{C}$ are conformally equivalent. Explain why Riemann insists on *proper subsets*.

Solution:

Assume that $\mathbb{C}$ is conformally equivalent to a proper subset R. Then by the Riemann mapping theorem R is conformal to D where D is the unit disc. Thus there is a conformal isomorphism $\phi : \mathbb{C} \to D$. But clearly $|\phi(z)| < 1$ and by Liouville's theorem ϕ is constant. This contradicts that ϕ is a conformal isomorphism. So no proper subset can be conformal to $\mathbb{C}$ itself.