Due Date: October 11, 2013 Friday NAME:..

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Math 302 Complex Analysis II – Homework 1 – Solutions

1	2	3	TOTAL
10	10	10	30

Please do not write anything inside the above boxes!

Check that there are 3 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Discuss the convergence of the sum

$$\sum_{n=0}^{\infty} \binom{3n}{2n} x^n,$$

where $x \in \mathbb{R}$. Find the value of the sum in terms of x when the series converges.

Solution:

First we determine for which x this series converges. For this we start with the ratio test. Let

$$a_n(x) = \binom{3n}{2n} x^n = \frac{(3n)!}{(2n)!n!} x^n, \; n = 0, 1, 2, \ldots.$$

Using the ratio test we have

$$\lim_{n\to\infty} \frac{a_{n+1}(x)}{a_n(x)} = \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n+3)}{(2n+1)(2n+2)(n+2)} |x| = \frac{27}{4} |x|.$$

Therefore the series converges absolutely for $|x| < \dfrac{4}{27}$.

We now check convergence at the end points. For $x = 4/27$ we use the Raabe test.

$$\lim_{n\to\infty} n \left(1 - \frac{a_{n+1}(4/27)}{a_n(4/27)} \right) = \frac{1}{2} < 1,$$

so the series diverges at $x = 4/27$.

To check the convergence at $x = -4/27$ we first notice that the series becomes an alternating series there so we need to understand the behavior of the general term $a_n(4/27)$. For this let

$$f(n) = \frac{a_{n+1}(4/27)}{a_n(4/27)}, \; n \geq 1,$$

where we consider n as a real parameter. Then since

$$f'(n) = \frac{2(9n^2 + 10n + 3)}{9(2n+1)^2(n+1)^2} > 0, \text{ and } \lim_{n\to\infty} f(n) = 1,$$

we have

$$f(n) < 1 \text{ and hence } a_n(4/27) > a_{n+1}(4/27) \text{ for all } n \geq 1.$$

This shows that the general term $a_n(4/27)$ is decreasing. Next we need to check if it decreases down to zero. For this we use Stirling's formula:

$$\lim_{n\to\infty} \frac{n!\, e^n}{\sqrt{2\pi}\, n^n\, n^{1/2}} = 1.$$

Let $S(n) = \dfrac{n!\, e^n}{\sqrt{2\pi}\, n^n\, n^{1/2}}$. Then $n! = S(n) \left[\dfrac{\sqrt{2\pi}\, n^n\, n^{1/2}}{e^n} \right]$. Using this we have

$$\lim_{n\to\infty} a_n(4/27) = \frac{1}{\sqrt{2\pi}} \left(\frac{3}{2} \right)^{1/2} \lim_{n\to\infty} \frac{S(3n)}{S(2n)S(n)} \lim_{n\to\infty} \left(\frac{1}{n} \right)^{1/2} = 0.$$

Hence by the alternating series test, the series also converges at $x = -4/27$. Thus the interval of convergence is

$$-\frac{4}{27} \leq x < \frac{4}{27}.$$

To calculate the series, we start with the usual observation that for any $R > 0$,

$$\binom{3n}{2n} = \frac{1}{2\pi i} \int_{|z|=R} \frac{(z+1)^{3n}}{z^{2n+1}}\, dz.$$

Then for some $R > 0$ to be determined later we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{3n}{2n} x^n &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|z|=R} \frac{(z+1)^{3n} x^n}{z^{2n+1}}\, dz \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|z|=R} \left[\frac{(1+z)^3 x}{z^2}\right]^n \frac{1}{z}\, dz \\
&= \frac{1}{2\pi i} \int_{|z|=R} \sum_{n=0}^{\infty} \left[\frac{(1+z)^3 x}{z^2}\right]^n \frac{1}{z}\, dz \\
&= \frac{1}{2\pi i} \int_{|z|=R} \frac{z}{z^2 - (1+z)^3 x}\, dz.
\end{aligned}$$

The change of infinite sum and integration is justified when the convergence of the infinite sum is uniform on the given circle. In particular we want to find an $R > 0$ such that

$$\left|\frac{(1+z)^3 x}{z^2}\right| < 1 \text{ for } |z| = R.$$

For this, first note that for $|z| = R$ and for $|x| < 4/27$ we have

$$\left|\frac{(1+z)^3 x}{z^2}\right| < \frac{4(1+R)^3}{27R^2}.$$

Considering the right hand side as a function of $R > 0$, we find that its minimum occurs at $R = 2$ where it takes the value 1. In fact we check that when $|x| < 4/27$ and $|z| = 2$, we have

$$\left|\frac{(1+z)^3 x}{z^2}\right| \leq \frac{27}{4}|x| < 1,$$

so the series converges uniformly on the circle $|z| = 2$ for $|x| < 4/27$, and in that case the change of integral and the infinite sum is justified.

Using residue theory at this point we conclude that for $|x| < 4/27$,

$$\sum_{n=0}^{\infty} \binom{3n}{2n} x^n = \sum_{z_i} \mathrm{Res}\left(\frac{z}{z^2 - (1+z)^3 x}, z_i\right),$$

where the sum is over all residues with $|z_i| < 2$. Since the denominator is a cubic whose roots are not readily available, we need to do some extra work to estimate the sum.

First note that when $x = 0$, the infinite sum collapses and its sum is 1. So in the following arguments we treat the cases when $x \neq 0$.

Now fix x and define a function

$$f(z) = x(z+1)^3 - z^2.$$

First note that the discriminant of this polynomial is $-x(27x-4)$, so the roots will be distinct for $x \neq 0, 4/27$.

Treating z as a real variable, we see that $f(z)$ always has at least one real root. Checking the maximum and minimum values of $f(z)$, again treating z as real, we see that $f(z)$ has two more real roots when $0 < x < 4/27$, and two imaginary roots when $-4/27 < x < 0$. Further analysis will reveal that in either case only two roots will have modulus less than 2. Let z_1 and z_2 be these two roots.

Recalling how the residue of a rational function with simple roots is calculated we have

$$\sum_{n=0}^{\infty} \binom{3n}{2n} x^n = \sum_{i=1}^{2} \frac{z_i}{2z_i - 3x(1+z_i)^2},$$

when $-4/27 \leq x < 4/27$ and $x \neq 0$. The equality at $x = -4/27$ follows as usual from Abel's theorem.

The above discussion reveals that for this infinite sum we need to find the roots of $x(z+1)^3 - z^2 = 0$ numerically and then use the above residue formula to estimate the sum. The power of this method is that roots of a cubic are easily found on the computer but the actual sum takes too long. The shortcoming of the method is that we could not find an analytic formula which directly depends on x and is executed immediately without any intermediate steps.

The convergence of the series is particularly slow when $|x|$ approaches $4/27 = 0.1481....$ In these cases the above algorithm is much faster than trying to add up the series term by term until a satisfactory result is obtained.

To fully appreciate the power of this formula take $x = -4/27$. The roots of $x(z+1)^3 - z^2 = 0$ are -9.44353 and $-0.15323 \pm i0.28707$. The formula gives the sum as 0.73784. On the other hand the 20000th term of the series is still 0.00345 so the second digit after the decimal point is not yet stabilized at this term.

Here is what I learned from **Burcu Özcan**'s solution set:

Let s be the value of the infinite sum once x is fixed. We know that s is always positive; either $x \geq 0$ and the sum is obviously positive or $x < 0$ in which case the sum is alternating and is starting with 1 with strictly decreasing terms. Hence in either case $s > 0$.

The function $\dfrac{z}{z^2 - x(1+z)^3}$ has no singularity at infinity, and has two singularities inside the circle $|z| < 2$, and one positive root $\lambda > 2$. Hence the sum of the residues at the singularities inside the circle $|z| < 2$ is equal to -1 times the residue at λ. Keep in mind that $\lambda^2 - x(1+\lambda)^3 = 0$.

The above argument yields $s = \dfrac{\lambda}{3x(1+\lambda)^2 - 2\lambda}$, and after simplifying

$$s = \frac{1+\lambda}{\lambda - 2}, \text{ or equivalently } \lambda = \frac{1+2s}{s-1}.$$

Substituting this value of λ into the equation it satisfies we find that s satisfies

$$(4 - 27x)s^3 - 3s - 1 = 0.$$

For the values $-4/27 \le x < 4/27$, this equation has one positive root and either two negative roots (when $x \ge 0$) or two complex conjugate roots (when $x < 0$). Hence the positive root of this equation is the required value of the infinite sum corresponding to the chosen x.

For example we can now give an exact algebraic value for s corresponding to $x = -4/27$ where the convergence is slowest. In fact we can now show off using this new idea.

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{3n}{2n} \left(\frac{-4}{27}\right)^n &= \frac{1}{4}(4+2\sqrt{2})^{1/3} + \frac{1}{2}(4+2\sqrt{2})^{-1/3} \approx 0.73784 \\
\sum_{n=0}^{\infty} \binom{3n}{2n} \left(\frac{-3}{27}\right)^n &= \frac{1}{14}(196+28\sqrt{21})^{1/3} + 2(196+28\sqrt{21})^{-1/3} \approx 0.781849 \\
\sum_{n=0}^{\infty} \binom{3n}{2n} \left(\frac{-2}{27}\right)^n &= \frac{1}{6}(18+6\sqrt{3})^{1/3} + (18+6\sqrt{3})^{-1/3} \approx 0.836243 \\
\sum_{n=0}^{\infty} \binom{3n}{2n} \left(\frac{-1}{27}\right)^n &= \frac{1}{10}(100+20\sqrt{5})^{1/3} + 2(100+10\sqrt{5})^{-1/3} \approx 0.905958 \\
\sum_{n=0}^{\infty} \binom{3n}{2n} \left(\frac{1}{27}\right)^n &= \frac{2}{\sqrt{3}} \cos\frac{\pi}{18} \approx 1.137158 \\
\sum_{n=0}^{\infty} \binom{3n}{2n} \left(\frac{2}{27}\right)^n &= \cos\frac{\pi}{3} \approx 1.366025 \\
\sum_{n=0}^{\infty} \binom{3n}{2n} \left(\frac{3}{27}\right)^n &= 2\cos\frac{\pi}{9} \approx 1.879385.
\end{aligned}$$

Have fun!

NAME: STUDENT NO:

Q-2) For each non-negative integer m, let

$$S_m = \sum_{n=1}^{\infty} \frac{1}{n^2 + m^2}.$$

Evaluate S_m.

Solution:

When $m = 0$, we know that the sum is $\dfrac{\pi^2}{6}$.

Now assume that $m \geq 1$.

Let $f(z) = \dfrac{1}{z^2 + m^2}$. The function $\phi(z) = f(z)\pi \cot \pi z$ has simple singularities at integer points and also at $z = \pm im$. The residues of $f(z)\pi \cot \pi z$ at these singularities add up to zero, as discussed in the textbook. Hence

$$0 = \sum_{n=-\infty}^{\infty} f(n) + \operatorname{Res}(\phi(z), z = im) + \operatorname{Res}(\phi(z), z = -im) = 2\sum_{n=1}^{\infty} \frac{1}{n^2 + m^2} + \frac{1}{m^2} - \frac{\pi}{m} \coth \pi m.$$

Finally we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + m^2} = \frac{1}{2}\left(\frac{\pi}{m} \coth \pi m - \frac{1}{m^2}\right).$$

Note that taking the limit of both sides as m tends to zero gives the correct answer $\pi^2/6$.

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Q-3) Let

$$I(n,\epsilon) = \int_{-i\infty}^{+i\infty} \frac{e^{\epsilon z}}{(z+\epsilon)^n},$$

where the integration is taken along the imaginary axis, n is a positive integer and $\epsilon \in \{-1,+1\}$. Evaluate $I(n,\epsilon)$.

Solution:

The residue of $\dfrac{e^{\epsilon z}}{(z+\epsilon)^n}$ at $z=-\epsilon$ is $\dfrac{\epsilon^{n-1}}{e\,(n-1)!}$. Hence

$$I(n,\epsilon) = \epsilon^{n-1}\frac{2\pi i}{e\,(n-1)!}.$$