Due Date: December 13, 2013 Friday NAME:..

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Math 302 Complex Analysis II – Homework 3 – Solutions

1	2	3	4	TOTAL
10	10	10	10	40

Please do not write anything inside the above boxes!

Check that there are 3 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Let f be an entire function and suppose that there is a positive integer n and positive real numbers A and R such that for all $z \in \mathbb{C}$ with $|z| \geq R$, we have $|f(z)| \leq A|z|^n$. Use Cauchy Integral Formula to show that f is a polynomial of degree at most n.

Solution:

For any $z_0 \in \mathbb{C}$ and for any $R_0 > 0$, the Cauchy Integral Formula, in generalized form, gives

$$f^{(n+1)}(z_0) = \frac{(n+1)!}{2\pi i} \int_{|z-z_0|=R_0} \frac{f(z)}{(z-z_0)^{n+2}}\, dz.$$

Let

$$C_{R_0} = \{z \in \mathbb{C} \mid |z - z_0| = R_0\}.$$

Fix any $z_0 \in \mathbb{C}$. Choose any $R_0 > |z_0| + R$. This choice of R_0 guarantees that for all $z \in C_{R_0}$ we have $R < |z| \leq |z_0| + R$. Hence in this case for all $z \in C_{R_0}$, we have $|f(z)| < A(|z_0| + R_0)^n$. Putting this into the Cauchy Integral Formula, we get

$$|f^{(n+1)}(z_0)| \leq \frac{(n+1)! R_0 (|z_0| + R_0)^n}{R_0^{n+2}}.$$

The left hand side is independent of R_0. The right hand side holds for all $R_0 > |z_0| + R$, and goes to zero as R_0 goes to infinity. This forces the left hand side to be zero to begin with. Hence $f^{(n+1)}(z_0) = 0$ for all $z_0 \in \mathbb{C}$, and consequently f is a polynomial of degree at most n.

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Q-2) Let $f = u + iv$ be an entire function. Theorem 16.10 says that if $|u(z)| \leq A|z|^n$ for all sufficiently large z, and for some constant $A > 0$ and for some non-negative integer n, then f is a polynomial of degree at most n. The proof uses Theorem 16.9 which says that if f is C-analytic in $D(0, R)$, for some $R > 0$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \, d\theta + iv(0).$$

Assuming that differentiation with respect to z can be carried inside the integral sign for all orders, give an alternate proof of Theorem 16.10 by showing that $f^{(n+1)}(z_0) = 0$ for all $z_0 \in \mathbb{C}$.

Solution:

Fix any $z_0 \in \mathbb{C}$ and choose any $R > 2|z_0|$. Then $|Re^{i\theta} - z_0| \geq R - |z_0| > R - R/2 = R/2$, and moreover

$$f^{(n+1)}(z_0) = \frac{R}{\pi} \int_0^{2\pi} u(Re^{i\theta}) \frac{e^{i\theta} n!}{(Re^{i\theta} - z_0)^{n+2}} \, d\theta.$$

It then follows that

$$|f^{(n+1)}(z_0)| \leq \frac{n! 2^{n+2} A}{\pi} \frac{1}{R},$$

for all $R > 2|z_0|$. This can only happen when $f^{(n+1)}(z_0) = 0$.

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Q-3) Let $u(x, y)$ and $v(x, y)$ be harmonic functions on a region D in $\mathbb{C}$. Find conditions on u and v such that uv is harmonic on D. Show that these conditions hold if $u + iv$ is analytic on D. Show however that when uv is harmonic, it does not necessarily imply that $u + iv$ is analytic on D.

Solution:

Let Δ be the laplace operator. Then $\Delta(uv) = \Delta u + \Delta v + 2(u_x v_x + u_y v_y)$.

It follows that when u and v are harmonic, uv will be harmonic if and only if $u_x v_x + u_y v_y = 0$. When $u + iv$ is analytic, the Cauchy-Riemann equations on u and v forces $u_x v_x + u_y v_y = 0$ but it is not sufficient. For example if $u = x$, $v = -y$, then uv is harmonic in the plane but $x - iy$ is not analytic anywhere.

NAME: STUDENT NO:

Q-4) Find the Weierstrass product form of the entire function $\sinh z$.

Solution:

$$\sinh z = z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2\pi^2}\right).$$

We can obtain this directly from the Weierstrass product form of $\sin z$ as follows:

$$\sinh z = -i \sin iz = (-i)(iz) \prod_{k=1}^{\infty} \left(1 - \frac{(iz)^2}{k^2\pi^2}\right) = z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2\pi^2}\right).$$

It is also a good exercise to construct the function $z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2\pi^2}\right)$ which vanishes only at the zeros of $\sinh z$ and then following the standard arguments of the book to show that it is actually equal to $\sinh z$.