Due Date: December 27, 2013 Friday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 302 Complex Analysis II – Homework 4 – Solutions

1	2	3	4	TOTAL
10	10	10	10	40

Please do not write anything inside the above boxes!

Check that there are 4 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Let f be an entire function of finite order with finitely many zeros. Show that either $f(z)$ is a polynomial or $f(z) + z$ has infinitely many zeros.

Solution:

If $f(z)$ is a polynomial, then we are done. If $f(z)$ is not a polynomial, then we know that $f(z) = P(z)e^{Q(z)}$ where P and Q are polynomials and $Q(z)$ is not constant. Suppose that $g(z) = f(z) + z$ has finitely many zeros. Since g is entire and is of finite order, it must be of the form

$$g(z) = R(z)e^{S(z)},$$

where R and S are polynomials. This give the equality

$$z + P(z)e^{Q(z)} = R(z)e^{S(z)}. \qquad (*)$$

Taking the second derivatives of both sides and rearranging we obtain an equality of the form

$$P_0(z)e^{Q(z)} = R_0(z)e^{S(z)},$$

where $P_0(z)$ and $R_0(z)$ are polynomials. This gives

$$e^{Q(z)-S(z)} = \frac{R_0(z)}{P_0(z)}.$$

Since the LHS has neither zeros nor poles, the RHS being a rational function of z must be constant. This implies in particular that $S(z) = Q(z)+c_0$, where $c_0 \in \mathbb{C}$ is a constant. Putting this into equation $(*)$, we get

$$e^{Q(z)} = \frac{z}{R(z)e^{c_0} - P(z)}.$$

A similar argument as above forces $Q(z)$ to be a constant, which is a contradiction.

Hence $f(z) + z$ must have infinitely many zeros.

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Q-2) Show that $\Gamma(z) = \lim_{n\to\infty} \int_0^n t^{z-1} \left(1 - \frac{t}{n}\right)^n dt$.

Solution:

This is an Exercise in Chapter 18, with hints on the back of the book. Here are the main steps involved in the solution.

From the Taylor expansion of e^x we first get

$$e^{-t/n} - (1 - \frac{t}{n}) = \frac{t^2}{2!n^2} - \frac{t^3}{3!n^3} + \cdots \leq \frac{t^2}{2!n^2}, \tag{1}$$

when $t < n$. Next we observe that when $0 < b < a$, we have for any positive integer n,

$$a^n - b^n = (a-b)(a^{n-1} + \cdots + a^{n-k-1}b^k + \cdots + b^{n-1}) \leq (a-b)\, n\, a^{n-1}. \tag{2}$$

Setting $a = e^{-t/n}$, $b = (1 - t/n)$, and noting that in this case $0 < b < a$ holds for any positive integer n, we get

$$e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \left(e^{-t/n} - (1 - \frac{t}{n})\right) n\, e^{-(t/n)(n-1)} \leq \frac{t^2 e^{-t} e}{2n},$$

where in the last step we combined the inequalities (1) and (2), and used the fact that $e^{t/n} < e$ when $t < n$. Finally we check that

$$\left| \int_0^n t^{z-1} e^{-t}\, dt - \int_0^n t^{z-1} \left(1 - \frac{t}{n}\right) dt \right| \leq \int_0^n t^{x-1} \left[e^{-t} - \left(1 - \frac{t}{n}\right)\right] dt,$$

where $z = x + iy$ and $x > 0$. We then have

$$\int_0^n t^{x-1} \left[e^{-t} - \left(1 - \frac{t}{n}\right)\right] dt \leq \frac{e}{2n} \int_0^n t^{(x+2)-1} e^{-t}\, dt \leq \frac{e}{2n} \Gamma(x+2).$$

We note that the last term tends to zero as n tends to infinity, thus showing the required identity.

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Q-3) Assume that $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} |z_k|^2$ converge. Show that $\prod_{k=1}^{\infty}(1+z_k)$ converges.

Solution:

This is an Exercise of Chapter 17, with hints at the back of the book.

We have

$$|\log(1+z_k) - z_k| \leq \frac{|z_k|^2}{2} + \frac{|z_k|^3}{3} + \cdots \leq |z_k|^2$$

when $|z_k| \leq 1/2$. So $\sum(\log(1+z_k) - z_k)$ converges (absolutely) when $\sum_{k=1}^{\infty} |z_k|^2$ converges. Since $\sum_{k=1}^{\infty} |z_k|$ also converges, $\sum \log(1+z_k)$ converges, which in turn implies that $\prod_{k=1}^{\infty}(1+z_k)$ converges.

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Q-4) Show that $\sum_{p \text{ prime}} \frac{1}{p}$ diverges.

Solution:

We have

$$\zeta(z) \prod_{p \text{ prime}} \left(1 - \frac{1}{p^z}\right) = 1, \ \Re z > 1.$$

Since $\lim_{z \to 1} \zeta(z) = \infty$, we must have $\prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)$ diverge to zero. Since $\sum 1/p^2$ converges, we must have $\sum 1/p$ diverge, which follows from the previous problem.