Math 302 Complex Analysis II – Midterm Exam 1 – Solutions

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Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

In this exam you are allowed to use two A4 size cheat-sheets provided that they are written by yourself, no photocopies are allowed. Your name must be written on both of them during the exam. You are asked to hand in your cheat-sheets together with your answers.

Q-1) Find the sum \( \sum_{n=0}^{\infty} \left( \frac{3n}{2n} \right) \left( \frac{12}{125} \right)^n \).

**Hint:** You may need to know that the roots of \( 125z^2 - 12(1 + z)^3 = 0 \) are approximately \(-\frac{1}{5}, \frac{2}{3}\) and 7.

**Solution:**

\[
\begin{align*}
\sum_{n=0}^{\infty} \left( \frac{3n}{2n} \right) \left( \frac{12}{125} \right)^n &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{C_R} \frac{(1 + z)^{3n}}{z^{2n}} \frac{12^n}{125^n} \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_{C_1} \left[ \frac{12(1 + z)^n}{125z^2} \right] \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_{C_1} \frac{125z}{125z^2 - 12(1 + z)^3} \frac{dz}{z} \\
&= \text{Res}(\frac{125z}{125z^2 - 12(1 + z)^3}, z = -1/5) + \text{Res}(\frac{125z}{125z^2 - 12(1 + z)^3}, z = 2/3) \\
&= \left( \frac{125z}{250z - 36(1 + z)^2} \right)_{z = -1/5} + \left( \frac{125z}{250z - 36(1 + z)^2} \right)_{z = 2/3} \\
&= \frac{5815}{3652} \approx 1.59.
\end{align*}
\]

The actual value, calculated using actual roots, is 1.60.
Q-2) For any integer \( n > 0 \), define

\[
I_n = \int_{2013-i\infty}^{2013+i\infty} \frac{2014^z}{z^{n+1}} \, dz.
\]

Calculate \( I_n \).

**Solution:**

\[
I_n = 2\pi i \text{Res}\left( \frac{2014^z}{z^{n+1}} , z = 0 \right)
\]

\[
= 2\pi i \frac{f^{(n)}(0)}{n!}, \quad \text{where } f(z) = 2014^z,
\]

\[
= 2\pi i \frac{1}{n!} (\ln 2014)^n.
\]

Here we used the principle branch in calculating \( 2014^z \). A general branch would be

\[
2014^z = \exp(z[\ln 2014 + 2\pi ik]), \quad \text{where } k \in \mathbb{Z}.
\]

Then the answer would be

\[
I_n = \frac{2\pi i}{n!} (\ln 2014 + 2\pi ik)^n,
\]

where \( k \) is an integer and \( k = 0 \) represents the principle branch.
Q-3)

(a) Calculate the cross-ratio \((i, 2i, 3, 4)\).

(b) Prove that the cross-ratio \((z_1, z_2, z_3, z_4)\) is real if and only if the four points \(z_1, z_2, z_3, z_4\) lie on a circle. (Here circle means a circle on the Riemann sphere so lines in the plane are also included in the meaning.)

Solution:

Recall that \((z_1, z_2, z_3, z_4) = \frac{z_4 - z_2}{z_4 - z_1} \frac{z_3 - z_1}{z_3 - z_2}\). Hence

\[
(i, 2i, 3, 4) = \frac{4 - 2i}{4 - i} \frac{3 - i}{3 - 2i} = \frac{210}{221} + \frac{10}{221}i.
\]

The transformation \(T(z) = \frac{z - z_2}{z - z_1} \frac{z_3 - z_1}{z_3 - z_2}\) sends \(z_1, z_2, z_3\) to \(\infty, 0, 1\) along the real line. We also know that Mobius transformations send circles to circles. Moreover \(T(z)\) lies on the same circle as \(T(z_1), T(z_2), T(z_3)\) if and only if \(T(z)\) is real, obviously! Hence \(z\) lies on the circle formed by \(z_1, z_2, z_3\) if and only if \(T(z)\) is real.
Q-4)

(a) Does there exist an automorphism $f$ of the unit disk $\mathbb{D}$ with the properties $f(0) = 0$ and $f'(0) = -3$?

(b) Let $f_1$ and $f_2$ be two conformal mappings of the upper half plane $\mathbb{H}$ onto the unit disk $\mathbb{D}$. Assume that for some point $z_0 \in \mathbb{H}$, we have $f_1(z_0) = f_2(z_0) = 0$ and $f_1'(z_0) > 0$, $f_2'(z_0) > 0$. Prove or disprove that $f_1 \equiv f_2$.

Solution:

If $f(0) = 0$, then $f(z) = e^{i\theta}z$ for some real $\theta$. Since $f'(0) = e^{i\theta} = -3$ has no solution in real $\theta$, such functions do not exist.

For the second part let $f(z) = f_2 \circ f_1^{-1}(z)$. Then $f$ is an automorphism of the unit disk with $f(0) = 0$ and $f'(0) = 1$. When $f(0) = 0$, we must have $f(z) = e^{i\theta}z$. Then $f'(0) = e^{i\theta} > 0$ forces $f(z) = z$. Hence $f_1 = f_2$ on $\mathbb{H}$. Observe that this argument works for any proper, non-empty, simply connected region $U$, and not just for $\mathbb{H}$. 
Q-5)  

(a) Let \( U \) be a proper, non-empty, simply connected region \( U \) in \( \mathbb{C} \). Notice that the Riemann Mapping Theorem makes no claim about the conformal equivalence of \( U \) with \( \mathbb{C} \). Complete this gap by proving or disproving that there exists a region \( U \) as above which is conformally equivalent to \( \mathbb{C} \).

(b) Let \( U \) be as above and let \( z_1, z_2 \) be two distinct points in \( U \). Prove or disprove that there exists a conformal map \( f : U \rightarrow U \) such that \( f(z_1) = z_2 \).

Solution:

Assume that there exists a conformal isomorphism \( f_1 : \mathbb{C} \rightarrow U \). By the Riemann Mapping Theorem there exists a conformal map \( f_2 : U \rightarrow \mathbb{D} \), where \( \mathbb{D} \) is the unit disk. Then \( f = f_2 \circ f_1 \) is a conformal isomorphism of \( \mathbb{C} \) onto the unit disk. But then for any \( z \in \mathbb{C} \), we must have \( |f(z)| < 1 \) and hence by Liouville's theorem \( f \) is constant and cannot be onto \( \mathbb{D} \). This contradiction shows that no such \( U \) exists.

For the second part let \( h \) be a conformal mapping of \( U \) onto the upper half plane \( \mathbb{H} \), whose existence is guaranteed by the Riemann Mapping Theorem. Define \( f : U \rightarrow U \) as \( f(z) = h^{-1}(h(z) - h(z_1) + h(z_2)) \). Then clearly \( f \) is a conformal map and \( f(z_1) = z_2 \). Here we used the fact that \( \mathbb{H} \) is closed under addition which is not something we can say about an arbitrary region \( U \).