Date: 30 December 2013, Monday
Time: 8:30-10:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 302 Complex Analysis II – Midterm Exam 2 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. **A correct answer without proper reasoning may not get any credit.**

In this exam you are allowed to use two A4 size cheat-sheets provided that they are written by yourself, no photocopies are allowed. Your name must be written on both of them during the exam. **You are asked to hand in your cheat-sheets together with your answers.**

Q-1) Show that $f(z) = \sum_{n=1}^{\infty} z^{n!}$ has a singularity at every point of its circle of convergence.

Solution:

By ratio test $\lim_{n\to\infty} \frac{z^{(n+1)!}}{z^{n!}} = \lim_{n\to\infty} z^{(n+1)!-n!} = 0$ if and only if $|z| < 1$. The circle of convergence is $|z| = 1$. Let ξ be a point on the unit disk such that $\xi^m = 1$ for some positive integer m. Let $n \geq m$. Then $\xi^{n!} = (\xi^m)^{n!/m} = 1$ since m divides $n!$. Then the general term $\xi^{n!}$ of the series becomes identically 1 for all $n \geq m$ and the series diverges at $z = \xi$. The points of the form $\xi^m = 1$ are dense on the unit circle, so the unit circle is a natural boundary for this series.

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Q-2) Show that

$$\frac{r^2}{2}\cos 2\phi + \frac{1}{2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\cos^2\theta)(1-r^2)}{1-2r\cos(\theta-\phi)+r^2}\, d\theta,$$

where $0 \le r < 1$ and $0 \le \phi < 2\pi$.

Solution:

Recall that the Poisson kernel in polar form is

$$\mathcal{K}(r,\phi;\theta) = \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}.$$

We know that there is a unique function $f(r,\phi)$ which is continuous on $r \le 1$, harmonic for $r < 1$, and agrees with the continuous function $u(e^{i\phi})$ when $r = 1$. Moreover, such an f is given by the relation

$$f(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\mathcal{K}(r,\phi;\theta)\, d\theta.$$

Here we have $f(r,\phi) = \frac{r^2}{2}\cos 2\phi + \frac{1}{2}$ which is harmonic everywhere and restricts to $\cos^2(\phi)$ on the unit disk. Hence it is the unique function claimed by the Poisson integral.

Recall that the Laplace operator in polar form is $\Delta f(r,\theta) = rf_r + r^2 f_{rr} + f_{\theta\theta}$

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Q-3) Find the value of the infinite sum $\displaystyle\sum_{m,n\geq 1} \frac{1}{m^2n^2}$.

Hint: $\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$.

Solution: The original solution I thought turned out to be wrong. My fault was to accept what the book wrote without checking it. The expression inside the box below is given wrong in the book. I give below the proof I had in mind using the infinite product expansion of $\sin \pi z$ and comparing it with its Taylor series.

$$\begin{aligned}
\sin \pi z &= \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \\
&= \pi z \left[1 - \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right) z^2 + \boxed{\left(\sum_{1\leq m<n} \frac{1}{m^2n^2}\right)} z^4 - \cdots \right] \\
&= \pi z - \frac{\pi^3 z^3}{6} + \frac{\pi^5 z^5}{120} - \cdots .
\end{aligned}$$

From this we see that

$$\sum_{m,n\geq 1} \frac{1}{m^2n^2} = 2 \sum_{1\leq m<n} \frac{1}{m^2n^2} + \sum_{k=1}^{\infty} \frac{1}{k^4} = 2 \cdot \frac{\pi^4}{120} + \frac{\pi^4}{90} = \frac{\pi^4}{36}.$$

Of course there is this easier solution which I learned from your papers.

$$\sum_{m,n\geq 1} \frac{1}{m^2n^2} = \sum_{m=1}^{\infty} \left(\frac{1}{m^2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}\right) = \sum_{m=1}^{\infty} \left(\frac{1}{m^2} \cdot \frac{\pi^2}{6}\right) = \frac{\pi^4}{36}.$$

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Q-4) Show that the Riemann zeta function $\zeta(z)$ has a simple pole with residue 1 at $z = 1$.

Hint: You may start with the identity $\Gamma(z) = n^z \int_0^\infty e^{-nt} t^{z-1}\, dt$, $n = 1, 2, \ldots$.

Solution:

$$\Gamma(z)\zeta(z) = \Gamma(z) \sum_{n=1}^{\infty} \frac{1}{n^z} = \int_0^\infty t^{z-1} \left(\sum_{n=1}^{\infty} e^{-nt} \right) dt = \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt.$$

Note that $\int_1^\infty \frac{t^{z-1}}{e^t-1}\, dt$ is an entire function. Moreover since

$$\frac{1}{e^t - 1} = \frac{1}{t} + A_0 + A_1 t + A_2 t^2 + \cdots ,$$

we have

$$\int_0^1 \frac{t^{z-1}}{e^t - 1}\, dt = \frac{1}{z-1} + \frac{A_0}{z} + \frac{A_1}{z+1} + \cdots ,$$

and hence

$$\Gamma(z)\zeta(z) = \frac{1}{z-1} + \frac{A_0}{z} + \frac{A_1}{z+1} + \cdots + \int_1^\infty \frac{t^{z-1}}{e^t - 1}\, dt.$$

The poles of $\Gamma(z)$ at $z = 0, -1, -2, \ldots$ cancel the poles of the RHS, and the only surviving pole is the one at $z = 1$ which is simple with residue 1.

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Q-5) Show that for a positive integer m the derivative of gamma function can be calculated as follows

$$\Gamma'(m+1) = m!\left(-\gamma + \sum_{k=1}^{m} \frac{1}{k}\right),$$

where γ is the usual Euler-Mascheroni constant.

Hint: You may start with the Weierstrass identity $\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}$.

Solution:

Take logarithm of both sides of the Weierstrass identity. Then take derivative of both sides and put $z = m + 1$. Next simplify the infinite sum and find

$$-\frac{\Gamma'(m+1)}{\Gamma(m+1)} = \gamma - \sum_{k=1}^{m} \frac{1}{k}.$$

The required identity now follows by observing that $\Gamma(m+1) = m!$.