Date: 13 May 2016, Friday
Time: 15:30-17:30
Instructor: Ali Sinan Sertöz

NAME:..

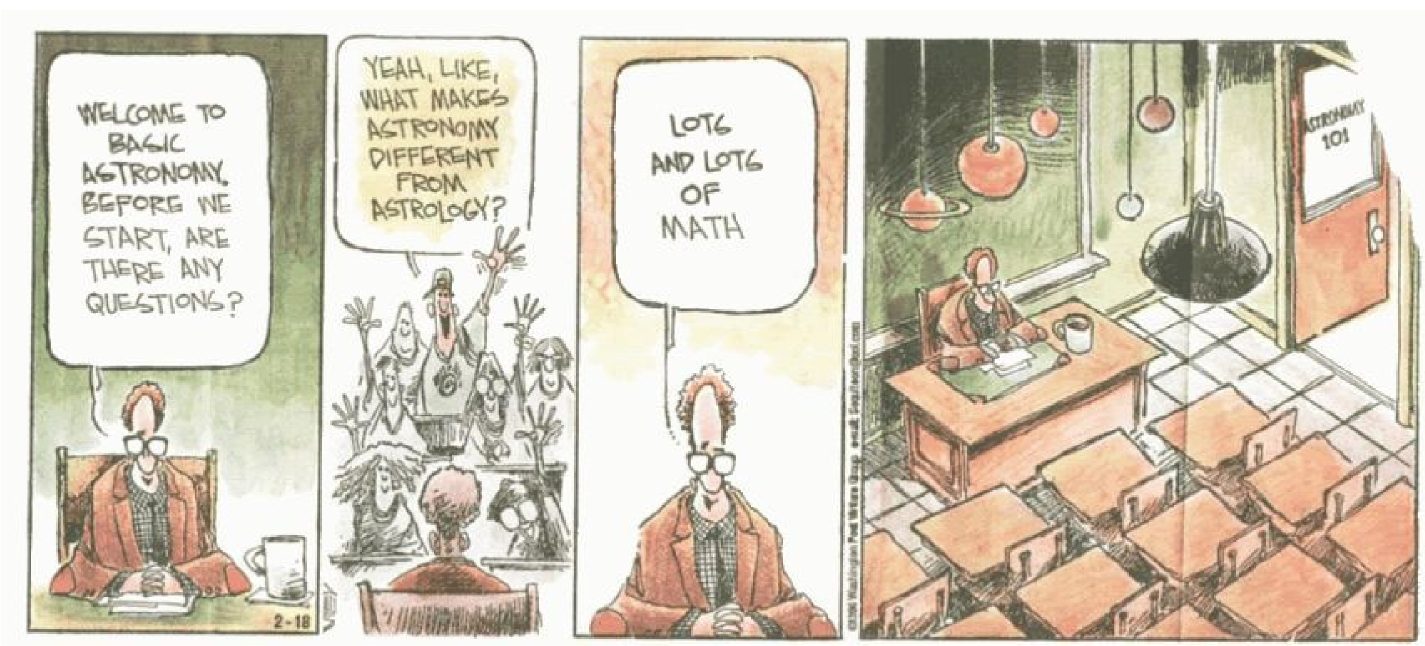
STUDENT NO:..

Math 302 Complex Analysis II – Final Exam – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.



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Q-1)

(a) Find a function f, analytic on $|z| < 1$ and such that $f(z) = 0$ if and only if $z = 1 - \frac{1}{k}$, where $k = 1, 2, 3, \ldots$.

(b) Find an entire function f such that $f(z) = 0$ if and only if $z = 1 - \frac{1}{k}$, where $k = 1, 2, 3, \ldots$.

Solution:

(a): By Weierstrass theory there exists an entire function $g(z)$ such that $g(z) = 0$ if and only if z is a positive integer. Then define $f(z) = g\left(\frac{1}{1-z}\right)$.

(b): The zero set has an accumulation point at $z = 0$, so such an f must be identically equal to zero.

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Q-2) Evaluate the integral $\displaystyle\int_{|z|=1/2} \frac{\cot(z)}{z^4 - z^5}\, dz$.

Hint: The Laurent series for the cotangent function is $\cot z = \frac{1}{z} - \frac{1}{3}\, z - \frac{1}{45}\, z^3 - \frac{2}{945}\, z^5 - \frac{1}{4725}\, z^7 - \frac{2}{93555}\, z^9 + \cdots$.

Solution:

The only singularity in the given circle is $z = 0$. Here we also have

$$\begin{aligned}
\frac{\cot(z)}{z^4 - z^5} &= \frac{1}{z^4}\,\frac{\cot z}{1-z} \\
&= \frac{1}{z^4}[\cot(z)\cdot(1 + z + z^2 + \cdots + z^n + \cdots)] \\
&= \frac{1}{z^4}[\frac{1}{z} + 1 + \frac{2}{3}z + \frac{2}{3}z^2 + \frac{29}{45}z^3 + \frac{29}{45}z^4 + \frac{607}{945}z^5 + \cdots] \\
&= \cdots + \frac{29}{45z} + \cdots
\end{aligned}$$

Hence the value of the integral is $2\pi i\,\frac{29}{45} = \frac{58\pi}{45}\, i \approx 4.049163866\, i$.

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Q-3) Recall that

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots, \quad \text{for} \quad \operatorname{Re} z > 1,$$

and that $\zeta(z)$ can be extended to the whole plane as a meromorphic function whose only singularity is a simple pole at $z = 1$ with residue 1.

Consider now the function

$$f(z) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots, \quad \text{for} \quad \operatorname{Re} z > 1.$$

Show that f can be extended to the whole plane analytically, i.e. with no poles.

Solution:

That $\zeta(z)$ has a simple pole at $z = 1$ with residue 1 means that

$$\lim_{z \to 1} (z-1)\zeta(z) = 1.$$

On the other hand the identity

$$f(z) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots = \left(1 - \frac{2}{2^z}\right)\zeta(z)$$

shows that f is analytic everywhere with the possible exception of $z = 1$. To check if f is analytic at $z = 1$ we consider the limit as $z \to 1$.

$$\lim_{z \to 1} f(z) = \lim_{z \to 1} \left(1 - \frac{2}{2^z}\right)\zeta(z) = \lim_{z \to 1} \left(\frac{2^z - 2}{2^z(z-1)}\right)((z-1)\zeta(z)) = \ln 2.$$

Existence of this limit implies that f is also analytic at $z = 1$.

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Q-4) Show that the unit circle is a natural boundary for the power series $\sum_{n=0}^{\infty} z^{n!}$.

Solution:

If α is a k-th root of unity then for any $n \geq k$ we have $\alpha^{n!} = 1$. Hence as z approaches α, the power series becomes an infinite sum of 1s, hence infinite. Since such α are dense on the boundary, no point on the boundary has an open neighborhood where the power series is finite.

On the other hand, using theorem 18.5, we see that $\frac{(n+1)!}{n!} \to \infty$ as $n \to \infty$. Since the limit is strictly larger than 1, the theorem concludes that he circle of convergence is a natural boundary.