Due Date: 28 March, Monday 2016
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:......................................

Math 302 Complex Analysis II – Homework 2 – Solutions

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework Assignments

(1) You may discuss the problems only with your classmates or with me. In particular you may not ask your assigned questions or any related question to online forums.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) It is absolutely mandatory that you write your answers alone. Any similarity with your written words and any other solution or any other source that I happen to know is a direct violation of honesty.

(4) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

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Q-1) Show that the cross-ratio of four distinct complex numbers is real-valued if and only if these four points lie on a circle or a line.

This is Exercise 15 on page 176 and there is a hint at the back of the book but the hint is incomplete. You must provide a full solution.

Solution:

The cross-ratio of four distinct points in the complex plane is where the fourth point is sent under the Mobius transformation which sends the first three points to 0, 1 and ∞, respectively. A Mobius transformation sends circles to circles, where a line is also considered as a circle since its stereographic pre-image is a circle on the Riemann sphere passing through the North pole. The points 0, 1 and ∞ already lie on the circle defined by the real line. So the image of the fourth point will lie on this circle, i.e. be real, if and only if it was already on the circle defined by the first three points.

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Q-2) Find a conformal map of the region between the two circles $|z| = 1$ and $|z - \frac{1}{4}| = \frac{1}{4}$ onto an annulus of the form $0 < \alpha < |z| < 1$.

This is Exercise 17 on page 177 and there is a hint at the back of the book but the hint is incomplete. You must provide a full solution.

Solution:

Let α be a complex number with $0 < |\alpha| < 1$. The Mobius transformation

$$T_1(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}$$

maps the unit disc onto the unit disc, sending α to zero.

On the other hand, the map $z \mapsto w = 4z - 1$ send the disc $|z - \frac{1}{4}| = \frac{1}{4}$ onto the unit disc. For any β inside the unit disc, the transformation $w \mapsto V = (w - \beta)/(1 - \bar{\beta}z)$ send the unit disc onto the unit disc, mapping β to zero. Finally, the map $V \mapsto \alpha V$ maps the unit disc onto the disc with radius $|\alpha|$, centered at the origin. Composing these maps we get the Mobius transformation

$$T_2(z) = \alpha\, \frac{4z - 1 - \beta}{1 - \bar{\beta}(4z - 1)},$$

which send the disc $|z - \frac{1}{4}| = \frac{1}{4}$ onto the annulus $0 < \alpha < |z| < 1$.

We now want to check if we can choose α and β, with $0 < |\alpha|, |\beta| < 1$ such that $T_1(z) = T_2(z)$. That common transformation now satisfies all the requirements of the problem. Setting $T_1(z)$ equal to $T_2(z)$ gives

$$(4|\alpha|^2 - 4\bar{\beta})z^2 + (4\alpha\bar{\beta} + 1 + \bar{\beta} - 4\alpha - |\alpha|^2 - |\alpha|^2\beta)z - (2i\alpha \operatorname{Im}\beta) = 0, \quad \text{for all } |z| < 1.$$

Since this polynomial has infinitely many roots, it must be identically zero. Hence all the coefficients are zero. From $(2i\alpha \operatorname{Im}\beta) = 0$ we get $\operatorname{Im}\beta = 0$ since $\alpha \neq 0$. So β is real. From $(4|\alpha|^2 - 4\bar{\beta}) = 0$ we get $\alpha\bar{\alpha} = \beta$. So $\beta > 0$. From $(4\alpha\bar{\beta} + 1 + \bar{\beta} - 4\alpha - |\alpha|^2 - |\alpha|^2\beta) = 0$ we get $(4\beta - 4)\alpha + (1 + \beta - |\alpha|^2 - |\alpha|^2\beta) = 0$, which shows that α is also real. After substituting α^2 for β, this last equation becomes

$$\alpha^4 - 4\alpha^3 + 4\alpha - 1 = 0.$$

Noticing that $\alpha = 1$ and $\alpha = -1$ are roots, we find that $\alpha^2 - 1$ divides this polynomial and we have

$$\alpha^4 - 4\alpha^3 + 4\alpha - 1 = (\alpha^2 - 1)(\alpha^2 - 4\alpha + 1) = 0,$$

which gives two more roots as $\alpha = 2 + \sqrt{3}$ and $\alpha = 2 - \sqrt{3}$. Among these four roots only $2 - \sqrt{3} \approx 0.267$ lies inside the unit disc. Hence the required Mobius transformation is

$$T(z) = \frac{z - 2 + \sqrt{3}}{1 - (2 - \sqrt{3})z}.$$

Note that we have

$$T(\alpha) = 0,\ T(1/2) = \alpha,\ T(0) = -\alpha.$$

Moreover the two fixed points of this transformation are on the unit circle, ± 1. If $z = c + is$ is a point on the unit circle, then $T(c + is)$ is also on the unit circle and in fact we have

$$T(c + is) = \frac{2c - 1}{2 - c} + i\frac{s(1 - \alpha^2)}{2\alpha(2 - c)}.$$

For special points, we notice that

$$T(e^{\pm\frac{\pi}{3}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}) = \pm i,\ T(\pm i) = e^{\pm\frac{2\pi}{3}} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2},\ T(e^{\pm\frac{2\pi}{3}}) = -\frac{4}{5} \pm i\frac{3}{5}.$$