Due Date: 11 April, Monday 2016
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:.......................................

Math 302 Complex Analysis II – Homework 3 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are **5** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Submit your solutions on this booklet only. Use extra pages if necessary.**

Rules for Homework Assignments

(1) You may discuss the problems only with your classmates or with me. In particular you may not ask your assigned questions or any related question to online forums.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) It is absolutely mandatory that you write your answers alone. Any similarity with your written words and any other solution or any other source that I happen to know is a direct violation of honesty.

(4) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. (It is a good thing to borrow ideas from friends but it is a bad thing not to acknowledge their contribution!)

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

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Q-1) Let A be a compact subset of $\mathbb{C}$. Assume that f_n is a sequence of analytic functions converging uniformly to a function f on A. Show that there exists a number $R > 0$ such that for all $z \in A$, we have $f_n(z)$ and $f(z)$ in the disk $D_R = \{z \in \mathbb{C} | \ |z| \leq R\}$.

This is one of the crucial details in the proof of Theorem 17.6 which is the main technical tool for the proof of Weierstrass Theorem, Theorem 17.7.

Solution:

Since the convergence is uniform, the limit function f is analytic hence continuous. Therefore $f(A)$ is compact. In particular there is $c > 0$ such that $|f(z)| < c$ for all $z \in A$. Moreover, uniform convergence also implies that there exists an integer $N > 0$ such that for all $n \geq N$ and for all $z \in A$, we have $|f_n(z) - f(z)| < 1$. This gives $|f_n(z)| \leq 1 + |f(z)| < 1 + c$. Moreover since each f_k is analytic, hence continuous, each of the sets $f_k(A)$ is bounded, say by R_k for $k = 1, 2, \ldots, N$. Let R be any number larger than $1 + c, R_1, \ldots, R_N$. This R establishes the claim.

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Q-2) Let X be a bounded region in $\mathbb{C}$. Show that e^z is uniformly continuous on X.

This is the other crucial detail in the proof of Theorem 17.6 which is the main technical tool for the proof of Weierstrass Theorem, Theorem 17.7.

Solution:

Choose an arbitrary $\epsilon > 0$. We want to show that there exists a $\delta > 0$ such that whenever we have $z_1, z_2 \in X$ with $|z_1 - z_2| < \delta$, we will have $|e^{z_1} - e^{z_2}| < \epsilon$. Recall that

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots, \quad \text{for all} \quad z \in \mathbb{C}.$$

Since X is bounded, there is $c > 0$ such that $|z| < c$ for all $z \in X$. Therefore for any integer $n > 1$ and for any $z_1, z_2 \in X$, we can write

$$|z_1^n - z_2^n| = |(z_1 - z_2)(z_1^{n-1} + z_1^{n-2}z_2 + \cdots + z_2^{n-1}| \leq |z_1 - z_2|\, nc^{n-1}.$$

We can now write

$$\begin{aligned}
|e^{z_1} - e^{z_2}| &= \left|(z_1 - z_2) + \frac{z_1^2 - z_2^2}{2!} + \frac{z_1^3 - z_2^3}{3!} + \cdots + \frac{z_1^n - z_2^n}{n!} + \cdots\right| \\
&\leq |z_1 - z_2| + \frac{|z_1^2 - z_2^2|}{2!} + \frac{|z_1^3 - z_2^3|}{3!} + \cdots + \frac{|z_1^n - z_2^n|}{n!} + \cdots \\
&\leq |z_1 - z_2| + |z_1 - z_2|\frac{c}{1!} + |z_1 - z_2|\frac{c^2}{2!} + \cdots + |z_1 - z_2|\frac{c^{n-1}}{(n-1)!} + \cdots \\
&= |z_1 - z_2|\left(1 + c + \frac{c^2}{2!} + \cdots + \frac{c^n}{n!} + \cdots\right) \\
&= |z_1 - z_2|\, e^c.
\end{aligned}$$

Now choosing $0 < \delta < \epsilon / e^c$ establishes the uniform continuity of e^z on X.

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Q-3) Let A be a compact set in $\mathbb{C}$, and let f_n be a sequence of analytic functions converging uniformly to a function f on A. Show that the sequence of functions $g_n(z) = e^{f_n(z)}$ converges uniformly to the function $g(z) = e^{f(z)}$ on A.

This should now follow from the results of questions 1 and 2.

Solution:

Start with an $\epsilon > 0$. Question 1 shows that there exists a bounded set X such that all $f_n(A)$ and $f(A)$ lie in X. The continuity of the exponential function shows that $g_n(z)$ converges to $g(z)$ at every point $z \in A$. Question 2 shows that the exponential function is uniformly continuous on X, so there exists a $\delta > 0$ such that whenever $|f_n(z) - f(z)| < \delta$, we will have $|g_n(z) - g(z)| < \epsilon$. Uniform continuity of g_n on A says that there then exists an integer N such that for all $n \geq N$ and for all $z \in A$ we have $|f_n(z) - f(z)| < \delta$. Putting these together we get the uniform convergence of g_n to g on A.

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Q-4) Show that any meromorphic function on $\mathbb{C}$ is the ratio of two entire functions.

You need to use Weierstrass Theorem.

Solution:

Let f be our meromorphic function. Weierstrass theorem says that there exists an entire function $h(z)$ whose zero set is exactly the pole set of f, the multiplicity of each zero being the same as the order of the pole. Then $g = fh$ is an entire function. Thus $f = g/h$ as claimed.

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Q-5) Show that $\displaystyle\sum_{1\le n_1<n_2<\cdots<n_k} \frac{1}{n_1^2\cdots n_k^2} = \frac{\pi^{2k}}{(2k+1)!}$.

Check how $\sum 1/n^2$ is calculated using the sine function and generalize.

Solution:

As an application of Weierstrass theorem we have $\displaystyle\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) = \frac{\sin \pi z}{\pi z}$. Hence we can write

$$\begin{aligned}
\sin \pi z &= \pi z \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) \\
&= \pi z\left[1-\left(\sum_{1\le n_1}\frac{1}{n_1^2}\right)z^2+\left(\sum_{1\le n_1<n_2}\frac{1}{n_1^2 n_2^2}\right)z^4-\cdots+(-1)^k\left(\sum_{1\le n_1<\cdots<n_k}\frac{1}{n_1^2\cdots n_k^2}\right)z^{2k}+\cdots\right]
\end{aligned}$$

But we also have from taylor expansion

$$\sin \pi z = \pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \cdots + (-1)^k \frac{\pi^{2k+1} z^{2k+1}}{(2k+1)!} + \cdots .$$

Comparing these two expressions for sine we get the claimed result.

In fact using Maple we found

$$\sum_{1\le n_1<n_2}^{n_2\le 2500} \frac{1}{n_1^2\, n_2^2} \approx 0.8110846632, \quad \text{and} \quad \frac{\pi^4}{5!} \approx 0.8117424256.$$

$$\sum_{1\le n_1<n_2<n_3}^{n_3\le 700} \frac{1}{n_1^2\, n_2^2\, n_3^2} \approx 0.1895946964, \quad \text{and} \quad \frac{\pi^6}{7!} \approx 0.1907518243.$$