Due Date: 20 May 2015, Wednesday
Due Time: 15:30
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:......................................

Math 430 / Math 505 Introduction to Complex Geometry – Final Exam – Solutions

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Submit your solutions on this booklet only. Use extra pages if necessary.**

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you write your answers alone. Any similarity with your written words with any other solution or any other source that I happen to know is a direct violation of honesty.

(2) In particular do not lend your written solutions to your friends, nor borrow your friends's written solutions. Oral exchange of ideas is acceptable and is in fact encouraged.

(3) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source can be easily retrieved by the reader. This includes any ideas you borrowed from your friends.

(4) Finally, in your written solution make sure that you exhibit your total understanding of the ideas involved, even mentioning where you quote a result but don't really follow the reasoning. This is an essential ingredient of learning.

Affidavit of compliance with the above rules: *I affirm that I have complied with the above rules in preparing this submitted work. Every solution I wrote reflects my true understanding of the problem. Any sources used, ideas from friends or others are explicitly cited without exception.*

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Q-1) Let $T_{n,d}$ be the space of all hypersurfaces of degree d in $\mathbb{P}^n$. Let $N_{n,d} = \dim T_{n,d}$. ($n > 1,\ d > 0$.)

1) Calculate $N_{n,d}$.

Let G_n be the space of all projective lines in $\mathbb{P}^n$.

2) Calculate $\dim G_n$.

Let $I_{n,d}$ denote the incidence space of pairs $(L, X) \in G_n \times T_{n,d}$ such that $L \subset X$. Consider the projection maps $pr_1 : I_{n,d} \to G_n$ and $pr_2 : I_{n,d} \to T_{n,d}$.

3) Find the dimension of a generic fibre of pr_1.

4) What is the dimension of $I_{n,d}$.

5) Show that a generic hypersurface $X \in T_{n,d}$ contains no line when $d > 2n - 3$.

Solution:

Generic fibre means that you are considering $pr_1^{-1}(p)$ where p is chosen on some Zariski open subset of G_n. Also recall that Zariski open means the complement of the zero set of a collection of polynomials.

A hypersurface X of degree d in $\mathbb{P}^n$ is given as the zero set of a homogeneous polynomial of degree d in the variables $x_0, \ldots, x_n$ where $[x_0 : \cdots : x_n]$ are the homogeneous coordinates of $\mathbb{P}^n$. There are $\binom{n+d}{d}$ coefficients of such a polynomial. Multiplying the polynomial by a non-zero constant does not change X, so the points in $\mathbb{P}^N$ parametrize all such hypersurfaces where $N = \binom{n+d}{d} - 1$ Hence $N_{n,d} = N$.

The space of all projective lines in $\mathbb{P}^n$ is the Grassmannian space of all two dimensional vector subspaces in $\mathbb{C}^{n+1}$. Therefore $\dim G_n = \dim G(2, n+1) = 2(n-1)$.

A line L in $\mathbb{P}^n$ is given by a parametrization of the form

$$x_i = \alpha_{i1}s + \alpha_{i2}t, \quad \text{where} \quad [s : t] \in \mathbb{P}^1 \quad \text{and} \quad \alpha_{ij} \in \mathbb{C}.$$

Let $m_0, \ldots m_N$ be the set of all monomials in the x_i. A surface $X \in T_{n,d}$ is given by a polynomial of the form

$$f(x) = c_0 m_0 + \cdots + c_N m_N, \quad \text{where} \quad c_i \in \mathbb{C}.$$

To say that $L \subset X$ amounts to saying that

$$f(L) := f(\alpha_{01}s + \alpha_{02}t, \ldots, \alpha_{N1}s + \alpha_{N2}t) \equiv 0 \quad \text{for all} \quad [s : t] \in \mathbb{P}^1.$$

Note that

$$c_i m_i(\alpha_{01}s + \alpha_{02}t, \ldots, \alpha_{N1}s + \alpha_{N2}t) = c_i \sum_{j=0}^{d} k_{ij} s^j t^{d-j},$$

where $k_i \in \mathbb{C}$ depending on the monomial m_i. In particular if $L \subset X$, we have

$$f(L) = \sum_{\ell=0}^{d} g_\ell(c_0, \ldots, c_N) s^\ell t^{d-\ell} \equiv 0,$$

where the coefficients are polynomials in $c_0, \ldots, c_N$. All these $d+1$ coefficients, $g_0, \ldots, g_d$ must vanish for $L \subset X$. This means that given $L \in G_n$, we must choose X from the zero set $Z(g_0, \ldots, g_d) \subset \mathbb{P}^N$ so that $L \subset X$. Generically the dimension of this set is $N - d - 1$, which is the dimension of a generic fibre.

The dimension of $I_{n,d}$ is the sum of the dimension of G_n and the dimension of a generic fibre of pr_1. Therefore the dimension is given by

$$\dim I_{n,d} = 2(n-1) + N - d - 1 = N + (2n-3) - d.$$

When $d > 2n - 3$, then the dimension of $I_{n,d}$ is strictly less than the dimension of $T_{n,d}$. Note that a hypersurface X contains a line if and only if there is a pair of the form $(X, L) \in I_{n,d}$. In other words X must be in $pr_2(I_{n,d})$. But now the dimension consideration shows that $pr_2(I_{n,d})$ is included in a proper subspace of $T_{n,d}$. Hence a generic X, i.e. each of those outside the image of pr_2, contains no line at all.

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Q-2) We know that the Hodge conjecture holds for threefolds, but with rational coefficients. Here we will show that the Hodge conjecture fails with integer coefficients.
Let $X \in \mathbb{P}^4$ be a generic hypersurface of degree 6.

1) Show that $H^2(X, \mathbb{Z}) \cong H^2(\mathbb{P}^4, \mathbb{Z}) \cong \mathbb{Z}$.
2) Show that $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ and is generated by a class α which is of type $(2, 2)$.
3) Show that α is not the Poincare dual of an algebraic cycle.

Solution:

That $H^2(X, \mathbb{Z}) \cong H^2(\mathbb{P}^4, \mathbb{Z})$ follows directly from Lefschetz Hyperplane Section Theorem. From this isomorphism we know that $H^2(X, \mathbb{Z})$ is generated by $[h]$ which is the cohomology class of a hyperplane h in $\mathbb{P}^4$.

From Poincare duality we know that $H^4(X, \mathbb{Z}) \cong H_2(X, \mathbb{Z}) \cong \mathbb{Z}\, h$. It then follows that $\alpha \cdot [h] = PD(\alpha) \cdot h = 1$. Therefore α must be of type $(2, 2)$, since it evaluates to a non-zero value on a hyperplane. Here PD denotes the Poincare dual.

Now assume that $PD(\alpha) = \sum n_i [C_i]$, where the n_i are integers and the C_i are irreducible curves in X. Since $H^4(X, \mathbb{Z})$ is cyclic, we must have $PD(\alpha) = n[C]$ for some integer n and some curve C. From $PD(\alpha) \cdot h = 1$, we find that $n = 1$ and C is a line contained in X.

Now using the result of the previous question we know that X contains no such line. This contradiction shows that α is not algebraic, and hence the Hodge conjecture fails for X if we use integer coefficients.