Due Date: 14 April 2015, Tuesday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:......................................

Math 430 / Math 505 Introduction to Complex Geometry – Homework 2 – Solutions

1	2	3	4	5	TOTAL
35	35	30	0	0	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Submit your solutions on this booklet only. Use extra pages if necessary.**

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you write your answers alone. Any similarity with your written words with any other solution or any other source that I happen to know is a direct violation of honesty.

(2) In particular do not lend your written solutions to your friends, nor borrow your friends's written solutions. Oral exchange of ideas is acceptable and is in fact encouraged.

(3) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source can be easily retrieved by the reader. This includes any ideas you borrowed from your friends.

(4) Finally, in your written solution make sure that you exhibit your total understanding of the ideas involved, even mentioning where you quote a result but don't really follow the reasoning. This is an essential ingredient of learning.

Affidavit of compliance with the above rules: *I affirm that I have complied with the above rules in preparing this submitted work. Every solution I wrote reflects my true understanding of the problem. Any sources used, ideas from friends or others are explicitly cited without exception.*

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NAME: STUDENT NO:

Q-1) Does there exist a holomorphic function $F : \mathbb{C}^n \to \mathbb{C}$, where $n \geq 2$, such that $F(x) = 0$ if and only if $x = 0$? If *yes*, construct one such function. If *no*, explain why.

Solution:

Let $\mathfrak{p}$ be the ideal generated by the germ of F in the local ring $\mathcal{O}_n$ of holomorphic germs around the origin in $\mathbb{C}^n$. Then by Krull's Hauptidealsatz (Hartshorne p7, Theorem 1.11A) $\mathfrak{p}$ has height 1. The zero set of F being the origin has dimension zero. This dimension is also equal to the Krull dimension of the quotient ring $\mathcal{O}_n/\mathfrak{p}$. But by Theorem 1.8A of Hartshorne p6, the dimension of this ring is $n-1$, since dimension of $\mathcal{O}_n$ is n. This contradiction shows that no such F exists. And in fact we showed that the zero set of F must be $n-1$ if F is not a unit, such as an exponential function.

An easier way to see that no such F exists without going into general theory is to consider the meromorphic function $1/F$ which has an isolated singularity if F has an isolated zero. But by Hartog's theorem this singularity can be removed whereas the zero of F cannot be removed. This contradiction also shows that no such F can exist, without giving the exact dimension of the zero set of F.

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Q-2) Show that

$$H^p(\mathbb{P}^n, \Omega^q) = \begin{cases} \mathbb{C} & \text{if } p = q \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

To do this with Cech cohomology arguments is quite cumbersome but possible, if you use the standard covering of Pp^n as a Leray covering.

The cleanest way of determining these cohomology groups is to use Hodge decomposition. See Griffiths and Harris page 118.

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Q-3) Let V be a real vector space of dimension $2n$ with an almost complex structure J on it. Show that there exists vectors $v_1, \ldots, v_n \in V$ such that $\{v_1, \ldots, v_n, J(v_1), \ldots, J(v_n)\}$ is a basis for V.

Solution:

Let $v_1 \in V$ be a non-zero vector. We claim that v_1 and $J(v_1)$ are linearly independent. To show this assume that

$$av_1 + bJ(v_1) = 0, \quad \text{where} \quad a, b \in \mathbb{R}.$$

Applying J to both sides of this equation we get

$$aJ(v_1) - bv_1 = 0.$$

Multiply the first equation by a, the second equation by $-b$, and add the two new equations to get

$$(a^2 + b^2)v_1 = 0.$$

Since v_1 is not the zero vector, we must have $a^2 + b^2 = 0$, which means $a = b = 0$, proving that v_1 and $J(v_1)$ are linearly independent.

Now choose v_2 as a non-zero vector not in the span of v_1 and $J(v_1)$. We claim that $v_1, J(v_1), v_2, J(v_2)$ are linearly independent. For this let

$$av_1 + bJ(v_1) + cv_2 + dJ(v_2) = 0, \quad \text{where} \quad a, b, c, d \in \mathbb{R}.$$

By applying J to this equation we have

$$-bv_1 + aJ(v_1) - dv_2 + cJ(v_2) = 0.$$

Multiply the first equation by c, the second by $-d$, and add side by side to get

$$(ac + bd)v_1 + (bc - ad)J(v_1) + (c^2 + d^2)v_2 = 0,$$

which says that v_2 is in the span of v_1 and $J(v_1)$, which is a contradiction unless $c = d = 0$. But then the first equation gives a linear dependence between v_1 and $J(v_1)$, so we also must have $a = b = 0$. Thus these vectors are linearly independent.

Repeating this argument we conclude the proof.