Due Date: 21 April 2015, Tuesday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:...

STUDENT NO:.......................................

Math 430 / Math 505 Introduction to Complex Geometry – Homework 3 – Solutions

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you write your answers alone. Any similarity with your written words with any other solution or any other source that I happen to know is a direct violation of honesty.

(2) In particular do not lend your written solutions to your friends, nor borrow your friends's written solutions. Oral exchange of ideas is acceptable and is in fact encouraged.

(3) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source can be easily retrieved by the reader. This includes any ideas you borrowed from your friends.

(4) Finally, in your written solution make sure that you exhibit your total understanding of the ideas involved, even mentioning where you quote a result but don't really follow the reasoning. This is an essential ingredient of learning.

Affidavit of compliance with the above rules: *I affirm that I have complied with the above rules in preparing this submitted work. Every solution I wrote reflects my true understanding of the problem. Any sources used, ideas from friends or others are explicitly cited without exception.*

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NAME: STUDENT NO:

Q-1) Let V be a real vector space of dimension $2n$ equipped with an almost complex structure $J : V \to V$, with $J^2 = -id$.
Choose $x_1 \in V$ as a non-zero vector.
Having chosen $x_1, \dots, x_k \in V$, where $1 \leq k < n$, choose $x_{k+1} \in V$ as a non-zero vector not in the span of $x_1, J(x_1), \dots, x_k, J(x_k)$.
Then the ordering of the basis $x_1, J(x_1), \dots, x_n, J(x_n)$ defines an orientation on V. Show that this orientation is canonical. In other words show that if $y_1, \dots, y_n$ are chosen as described above, then the orientation defined by the ordered basis $y_1, J(y_1), \dots, y_n, J(y_n)$ is the same as the orientation defined above.

Solution:

Find the matrix of change of coordinates and show that its determinant is positive.

he change of bases matrix is a real $2n \times 2n$ matrix consisting of n^2 blocks each of which is of the form

$$\begin{matrix} a & b \\ -b & a \end{matrix}$$

By applying some row and column operations you can conclude that the determinant is positive.

NAME: STUDENT NO:

Q-2) Calculate the cohomology groups $H^p(\mathbb{P}^n, \mathcal{O})$ and $H^p(\mathbb{P}^n, \mathcal{O}^*)$ for all $p \geq 0$ and all $n \geq 1$.

Solution:

First one from Dolbeault theorem. The other from exponential sequence.

We know that $H^p(\mathbb{P}^n, \mathcal{O}) \cong H^{0,p}_{\bar\partial}(\mathbb{P}^n)$, and from page 118 of Griffiths and Harris we find that this group is $\mathbb{C}$ if $p = 0$, and is zero otherwise.

On the other hand, since the only global holomorphic functions on $\mathbb{P}^n$ are constants, the non-vanishing holomorphic ones are $\mathbb{C}^*$. Thus $H^0(\mathbb{P}^n, \mathcal{O}*) = \mathbb{C}^*$.

We can recover this and all other cohomology from the long exact sequence of cohomology associated to the short exact exponential sequence of sheaves.

First recall that

$$H^p(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p \text{ is even and } 0 \leq p \leq 2n, \\ 0 & \text{otherwise} \end{cases},$$

which follows from the cell decomposition of $\mathbb{P}^n$ which has cells only in even dimensions.

The above mentioned long exact sequence first gives

$$0 \to H^0(\mathbb{P}^n, \mathbb{Z}) \to H^0(\mathbb{P}^n, \mathcal{O}) \to H^0(\mathbb{P}^n, \mathcal{O}^*) \to 0,$$

which is nothing but

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{C} \stackrel{exp}{\to} H^0(\mathbb{P}^n, \mathcal{O}^*) \to 0,$$

from which we conclude that $H^0(\mathbb{P}^n, \mathcal{O}*) = \mathbb{C}^*$ as expected.

For $p > 0$ we consider the part

$$\cdots \to H^p(\mathbb{P}^n, \mathcal{O}) \to H^p(\mathbb{P}^n, \mathcal{O}^*) \to H^{p+1}(\mathbb{P}^n, \mathbb{Z}) \to H^{p+1}(\mathbb{P}^n, \mathcal{O}) goes \cdots .$$

We calculated above that all $H^k(\mathbb{P}^n, \mathcal{O}) = 0$ for $k > 0$, so we get the isomorphisms

$$H^p(\mathbb{P}^n, \mathcal{O}^*) \cong H^{p+1}(\mathbb{P}^n, \mathbb{Z}),$$

which gives

$$H^p(\mathbb{P}^n, \mathcal{O}^*) = \begin{cases} \mathbb{Z} & \text{if } p \text{ is odd and } 0 < p < 2n, \\ 0 & \text{otherwise} \end{cases}.$$