Due Date: 16 April 2015, Thursday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:...

STUDENT NO:.......................................

Math 430 / Math 505 Introduction to Complex Geometry – Midterm Exam I – Solutions

1	2	3	4	5	TOTAL
40	30	30	0	0	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Submit your solutions on this booklet only. Use extra pages if necessary.**

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me but it is absolutely mandatory that you write your answers alone. Any similarity with your written words with any other solution or any other source that I happen to know is a direct violation of honesty.

(2) In particular do not lend your written solutions to your friends, nor borrow your friends's written solutions. Oral exchange of ideas is acceptable and is in fact encouraged.

(3) You must obey the usual rules of attribution: all sources you use must be explicitly cited in such a manner that the source can be easily retrieved by the reader. This includes any ideas you borrowed from your friends.

(4) Finally, in your written solution make sure that you exhibit your total understanding of the ideas involved, even mentioning where you quote a result but don't really follow the reasoning. This is an essential ingredient of learning.

Affidavit of compliance with the above rules: *I affirm that I have complied with the above rules in preparing this submitted work. Every solution I wrote reflects my true understanding of the problem. Any sources used, ideas from friends or others are explicitly cited without exception.*

Please sign here:

NAME: STUDENT NO:

Q-1) Let $M \subset \mathbb{C}^n$ be a complex manifold of dimension $n-1$. Prove or disprove that for any point $p \in M$, there exists an open neighborhood $U \subset \mathbb{C}^n$ of p, and a holomorphic function f on U such that the zero set of f is precisely $U \cap M$.

Solution:

Let $p \in M \subset \mathbb{C}^n$ and $\mathcal{O}_p$ be the germs of holomorphic functions defined around p. i.e. each holomorphic function f defined on an open set $U \subset \mathbb{C}^n$, with $p \in U$, represents a germ. Let $\mathfrak{m}_p \subset \mathcal{O}$ be the ideal of germs vanishing at p. Assume that $\mathfrak{m}_p$ properly contains a non-trivial prime ideal $\mathfrak{p}$. The zero set N of $\mathfrak{p}$ is irreducible and properly contains the zero set of $\mathfrak{m}_p$. The latter is the germ represented by M. As germs we have

$$M \subsetneq N \subsetneq \mathbb{C}^n,$$

which entails

$$n-1 = \dim M < \dim N < \dim \mathbb{C}^n = n.$$

This leaves no integer for $\dim N$, so this contradiction shows that $\mathfrak{m}_p$ contains no non-trivial proper prime ideal. This means $\mathfrak{m}_p$ has height one.

We now quote two algebraic results.

1) $\mathfrak{m}_p$ is a unique factorization domain; see Griffiths and Harris p10.

2) A noetherian integral domain is a unique factorization domain if and only if every prime ideal of height one is principal; see Hartshorne p7.

At this point we now know that $\mathfrak{m}_p$ is principal, say it is generated by a germ which can be represented by a function f on U. Then $M \cap U$ is precisely the zero set of f.

For other approaches to the problem see Griffiths and Harris p13 (paragraph 3), or Huybrechts p21 remark 1.1.32. The algebraic case, i.e. when all holomorphic functions are taken to be polynomials, see Hartshorne p7 Proposition 1.13.

NAME: STUDENT NO:

Q-2) Let T be an elliptic curve. Show that any holomorphic map $f_n : \mathbb{P}^n \to T$ is constant, $n \geq 1$.

Solution:

First take $n = 1$. Let g denote the genus of a curve. We know that $g(\mathbb{P}^1) = 0$ and $g(T) = 1$.

Assume that f_1 is non-constant. By Hurwitz formula we have

$$2g(\mathbb{P}^1) - 2 = d(2g(T) - 2) + \deg R,$$

where $d > 0$ is the degree of f_1, and R is the ramification divisor which is an effective divisor. In particular $\deg R \geq 0$. We then have

$$-2 = \deg R \geq 0,$$

which is a contradiction showing that f_1 must be constant.

Now for the general case. Suppose f_n is non-constant, where $n > 1$.

Since f_n is non-constant, there exist two points $p, q \in \mathbb{P}^n$ such that $f_n(p) \neq f_n(q)$. Let $\mathbb{P}^1 \subset \mathbb{P}^n$ be the line joining the points p and q. Restricting f_n to this $\mathbb{P}^1$ we obtain a map from $\mathbb{P}^1$ to T which we showed to be constant. We then have $f_n(p) = f_n(q)$, which is a contradiction. So f_n must also be constant.

NAME: STUDENT NO:

Q-3) Using Čech cohomology techniques, show that for any meromorphic function ϕ on $\mathbb{C}$, there exist two entire functions f and g such that $\phi = f/g$. (In complex analysis we prove this using Weierstrass factorization theorem.)

Solution:

Let $\{U_\alpha\}$ be an open cover of $\mathbb{C}$ with open disks such that we can write

$$\phi|_{U_\alpha} = \frac{f_\alpha}{g_\alpha},$$

where f_α and g_α are holomorphic on U_α and have no common zeros. Define

$$h_{\alpha\beta} = \frac{f_\alpha}{f_\beta}.$$

Since any zero of f_α on $U_\alpha \cap U_\beta$ is a zero of ϕ with the same multiplicity, and since the same is true for f_β, we must have

$$h_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

Moreover if we set $h = \{h_{\alpha\beta}\}$, we see that

$$\delta(h)_{\alpha\beta\gamma} = h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta} = 1,$$

so h defines a cohomology class in $H^1(\mathbb{C}, \mathcal{O}^*)$. Here we use the fact that the covering we chose is Leray. Now since $H^1(\mathbb{C}, \mathcal{O}^*) = 0$, (see GH p47), there must be a 0-cochain $k = \{k_\alpha\} \in C^0(\{U_\alpha\}, \mathcal{O}^*)$ such that $\delta(k) = h$. This gives

$$\frac{k_\beta}{k_\alpha} = h_{\alpha\beta} = \frac{f_\alpha}{f_\beta},$$

which in turn gives

$$k_\alpha f_\alpha = k_\beta f_\beta.$$

Thus there exists an entire function F such that

$$F|_{U_\alpha} = k_\alpha f_\alpha.$$

Note that, since each $k_\alpha \in \mathcal{O}^*(U_\alpha)$, the entire function F has the same zeros as ϕ with the same multiplicities.

Similarly there exists an entire function G which has the same zeros of $1/\phi$ with the same multiplicities. Then the function H defined as

$$H = \phi \frac{G}{F}$$

is an entire function which has no zeros. Finally we see that

$$\phi = \frac{FH}{G},$$

as claimed.