Due Date: 26 May 2017, Friday
Time: until 17:30
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:.......................................

Math 430 / Math 505 Introduction to Complex Geometry – Final – Solutions

1	2	3	4	5	TOTAL
30	30	30	10	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me, or even with people who took this course before. It is not considered good behavior to ask these questions at online forums without mentioning that these are homework questions of an introductory course.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) **It is absolutely mandatory that you write your answers alone.**

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. *(It is always nice to flatter your friends by using their ideas and thanking them.)*

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

(6) Do not lend your written work to your friends and do not ask to borrow their written work. You may explain your solutions to your friends to any degree of detail you like, or you may ask them as many questions as they are willing to answer. **But the final writing process should be done alone.**

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work. Moreover I declare that every solution I wrote reflects my true understanding of the problem, and any sources used, including ideas from friends, are explicitly cited without exception.

Please sign here: *Date:*

NAME: STUDENT NO:

Q-1) Let s and s' be two holomorphic sections of a line bundle L on the complex manifold M. Show that s/s' is a meromorphic function on M.

Solution:

Let a local trivialization of L be given by $\{\phi_\alpha, U_\alpha\}$. Define the function s_α on U_α as follows: For a point $p \in U_\alpha$ we have $\phi_\alpha(s(p)) = (p, s_\alpha(p))$. Similarly define s'_α. Let the transition functions of L with respect to this trivialization be denoted by $(g_{\alpha\beta})$. For $p \in U_\alpha \cap U_\beta$ we have

$$s_\alpha(p) = g_{\alpha\beta}(p) s_\beta(p).$$

Similarly

$$s'_\alpha(p) = g_{\alpha\beta}(p) s'_\beta(p).$$

Now it is clear that on $U_\alpha \cap U_\beta$ we have

$$\frac{s_\alpha}{s'_\alpha} = \frac{s_\beta}{s'_\beta}.$$

This then defines a global meromorphic function on M.

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Q-2) Show that the hyperplane bundle $[H]$ on $\mathbb{P}^n$ always has a holomorphic section which vanishes exactly on the hyperplane H.

Solution:

Let $x_0, \ldots, x_n$ be the homogeneous coordinates on $\mathbb{P}^n$. Let U_i be the open set in $\mathbb{P}^n$ defined as the set where $x_i \neq 0$. This gives an open covering of $\mathbb{P}^n$. Let H be the hyperplane defined as $x_0 = 0$. Notice that $H \cap U_i$ is the divisor associated to the rational function x_0/x_i. Then the hyperplane bundle is defined by the transition functions x_i/x_j on $U_i \cap U_j$. Define a section s of $[H]$ as

$$s(x) = (x, \frac{x_0}{x_i}) \quad \text{if } x \in U_i.$$

Check that this defines a global section, using the transition functions. It is clear that it vanishes exactly on H.

NAME: STUDENT NO:

Q-3) Prove that every line bundle on a compact complex Riemann surface X is the line bundle associated to some divisor.

You can use the following data: For a line bundle L define

$$h^q(L) = \dim_{\mathbb{C}} H^q(X, \mathcal{O}(L)),\ q = 0, 1.$$

The problem is to show that $h^0(L) \neq 0$ for all line bundles L.

The Chern class $c_1(L)$ of a line bundle is an integer also denoted by $\deg(L)$. For two line bundles L_1 and L_2 we have $\deg(L_1 + L_2) = \deg(L_1) + \deg(L_2)$. Also $\deg(-L) = -\deg(L)$. If $D = \sum n_p \cdot p$ is a divisor, then we define $\deg(D) = \sum n_p$. Then $\deg([D]) = \deg(D)$. If $\deg(L) < 0$ then $h^0(L) = 0$. If we denote the canonical divisor of X by K, then $h^1(L) = h^0(K - L)$. Define a function on line bundles as

$$\phi(L) := h^0(L) - h^1(L) - \deg(L),$$

where L is a line bundle. It is known that this function is constant. It is also known that $\phi(L + [D]) = \phi(L)$ for any divisor D.

First fix an arbitrary line bundle L and show that if $h^0(L + [D]) \neq 0$ for some divisor D, then $h^0(L) \neq 0$. So assume that $h^0(L + [D]) = 0$ for all divisors D and obtain a contradiction.

Hint: Use $\phi(L) = \phi(L + [D])$ and show that the right hand side actually depends on D when $\deg(D)$ is very large.

Solution:

We know that every line bundle associated to a divisor has a meromorphic section which vanishes exactly on that divisor. Let s be such a section of $[D]$. If there is a non-zero element t in $H^0(X, L + [D])$, then t/s is a global section of L.

Assume now that $h^0(L+[D]) = 0$ for all divisors D. Choose a divisor D such that $\deg(K) - \deg(L) - \deg(D) < 0$. For this divisor we have

$$\begin{aligned}
\phi(L) &= \phi(L + [D]) \\
&= h^0(L + [D]) - h^1(L + [D]) - \deg(L + [D]) \\
&= -h^0(K - L - [D]) - \deg(L) - \deg(D) \\
&= -\deg(L) - \deg(D).
\end{aligned}$$

The right hand side depends on D and violates the fact that $\phi(L)$ is constant.

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Q-4) Speculate on how you would prove the Hodge conjecture.

Speculation: