Due Date: 17 February 2017, Thursday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:..
STUDENT NO:.......................................

Math 430 / Math 505 Introduction to Complex Geometry – Homework 1 – Solutions

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me, or even with people who took this course before. It is not considered good behavior to ask these questions at online forums without mentioning that these are homework questions of an introductory course.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) **It is absolutely mandatory that you write your answers alone.**

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. *(It is always nice to flatter your friends by using their ideas and thanking them.)*

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

(6) Do not lend your written work to your friends and do not ask to borrow their written work. You may explain your solutions to your friends to any degree of detail you like, or you may ask them as many questions as they are willing to answer. **But the final writing process should be done alone.**

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work. Moreover I declare that every solution I wrote reflects my true understanding of the problem, and any sources used, including ideas from friends, are explicitly cited without exception.

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Q-1) Let $\epsilon = (5, 13)$, $\delta = (2, 2)$ and $a = (a_1, a_2) = (3 + 4i, 5 + 12i) \in \mathbb{C}^2$. Define the polydisks $B_\epsilon(0)$ and $B_\delta(a)$ as

$$B_\epsilon(0) = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < 5, |z_2| < 13\}$$
$$B_\delta(a) = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1 - a_1| < 2, |z_2 - a_2| < 2\}$$

Find two points $p = (p_1, p_2) \in \mathbb{C}^2$ and $q = (q_1, q_2) \in \mathbb{C}^2$ such that $a \neq q$ and

$$p \in B_\delta(a) \cap B_\epsilon(0)$$
$$q \in B_\delta(a) \setminus B_\epsilon(0).$$

Also show that a is a boundary point of $B_\epsilon(0)$ in the sense that for any $\delta = (\delta_1, \delta_2)$ with $\delta_1, \delta_2 > 0$ there are points p and q such that

$$p \in B_\delta(a) \cap B_\epsilon(0)$$
$$q \in B_\delta(a) \setminus B_\epsilon(0).$$

Solution:

Check that the points $p = (2 + 4i, 4 + 12i)$ and $q = (4 + 4i, 4 + 12i)$ work for the first part.

For the second part let

$$u = \frac{a_1}{5} \quad \text{and} \quad v = \frac{a_2}{5}.$$

Define

$$p = (p_1, p_2) = \left(\left(5 - \frac{\delta_1}{2}\right) u, \left(13 - \frac{\delta_2}{2}\right) v\right)$$

and

$$q = (q_1, q_2) = \left(\left(5 + \frac{\delta_1}{2}\right) u, \left(13 + \frac{\delta_2}{2}\right) v\right)$$

These points satisfy the requirements. Note that $q = a$ also works in this case.

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Q-2) Let $P(z_1, z_2)$ be a non-constant polynomial with complex coefficients of two variables. Define the zero set $Z(P)$ of P as

$$Z(P) = \{(z_1, z_2) \in \mathbb{C}^2 \mid P(z_1, z_2) = 0\}.$$

Show that $Z(P)$ is not bounded. In other words show that for any number $R > 0$ there exists a point $(z_1, z_2) \in Z(P)$ with either $|z_1| \geq R$ or $|z_2| \geq R$.

Solution:

Choose any $R > 0$.

Let the degree of P as a polynomial in z_1 be n. Then

$$P(z_1, z_2) = p_n(z_2)z_1^n + \cdots + p_0(z_2),$$

where each $p_k(z_2)$ is a polynomial in z_2 of degree k with complex coefficients and $p_n(z_2) \not\equiv 0$. If $n = 0$ then $P = p_0(z_2)$ which is non-constant and must have a finite number of roots. Say a is one of them. Then take any $b \in \mathbb{C}$ with $|b| \geq R$. Then $(b, a) \in Z(P)$ as claimed.

Now assume $n > 0$. Since $p_n(z_2)$ is not identically zero, it must have only finitely many zeros, possibly none if it is a non-zero constant. In either case there is a point $a \in \mathbb{C}$ with $|a| \geq R$ such that $p_n(a) \neq 0$. Then $P(z_1, a)$ becomes a non-constant polynomial in z_1 with complex coefficients and must have at least one root say $b \in \mathbb{C}$. Then $(b, a) \in Z(P)$ as claimed.

In general we can prove that the zero set of a holomorphic function $P(z)$ defined on $\mathbb{C}^n$ is unbounded. Assume that $Z(P)$ is bounded and contained in a polydisc Δ. Then $1/P(z)$ is analytic and nonzero on $\mathbb{C}^n \setminus \Delta$. By Hartog's theorem we can extend $1/P(z)$ to an analytic function Q over Δ. Then $Q(z)$ is analytic on all of $\mathbb{C}^n$. This gives $Q(z)P(z) = 1$ on $\mathbb{C}^n$ which is a contradiction since P is not zero-free.