Due Date: 31 March 2017, Friday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:...

STUDENT NO:......................................

Math 430 / Math 505 Introduction to Complex Geometry – Homework 3 – Solutions

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me, or even with people who took this course before. It is not considered good behavior to ask these questions at online forums without mentioning that these are homework questions of an introductory course.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) **It is absolutely mandatory that you write your answers alone.**

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. *(It is always nice to flatter your friends by using their ideas and thanking them.)*

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

(6) Do not lend your written work to your friends and do not ask to borrow their written work. You may explain your solutions to your friends to any degree of detail you like, or you may ask them as many questions as they are willing to answer. **But the final writing process should be done alone.**

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work. Moreover I declare that every solution I wrote reflects my true understanding of the problem, and any sources used, including ideas from friends, are explicitly cited without exception.

Please sign here: *Date:*

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Q-1) Derive the complex chain rule. Here is the notation to use: $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are non-empty open subsets. Coordinates of $\mathbb{C}^n$ are $z = (z_1, \dots, z_n)$ where $z_j = x_j + iy_j$. Coordinates of $\mathbb{C}^m$ are $w = (w_1, \dots, w_m)$ with $w_j = s_j + it_j$. Let $f : U \to V$ be a C^∞-map with $f = (f_1, \dots, f_m)$ and each $f_j = u_j + iv_j$. Let $g : V \to \mathbb{C}$ be another C^∞-map. Derive formulas for

$$\frac{\partial}{\partial z_j}(g \circ f)(z) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j}(g \circ f)(z).$$

Your formulas should involve only complex differentiation naturally. Using your formula check if $g \circ f$ is holomorphic when f and g are. You can of course use the usual chain rule for real variables.

Solution:

Note first that $w_k = f_k$ means $s_k + it_k = u_k + iv_k$.

Also note that by definition

$$\frac{\partial}{\partial s_k} = \frac{\partial}{\partial w_k} + \frac{\partial}{\partial \bar{w}_k}, \quad \text{and} \quad \frac{\partial}{\partial t_k} = i\frac{\partial}{\partial w_k} - i\frac{\partial}{\partial \bar{w}_k}.$$

$$\begin{aligned}
\frac{\partial}{\partial z_j}(g \circ f)(z) &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(g \circ f)(z) \\
&= \frac{1}{2}\sum_{k=1}^{m}\left(\frac{\partial g}{\partial s_k}\frac{\partial u_k}{\partial x_j} + \frac{\partial g}{\partial t_k}\frac{\partial v_k}{\partial x_j} - i\frac{\partial g}{\partial s_k}\frac{\partial u_k}{\partial y_j} - i\frac{\partial g}{\partial t_k}\frac{\partial v_k}{\partial y_j}\right) \\
&= \sum_{k=1}^{m}\left(\frac{\partial g}{\partial s_k}\frac{1}{2}\left[\frac{\partial u_k}{\partial x_j} - i\frac{\partial u_k}{\partial y_j}\right] + \frac{\partial g}{\partial t_k}\frac{1}{2}\left[\frac{\partial v_k}{\partial x_j} - i\frac{\partial v_k}{\partial j_j}\right]\right) \\
&= \sum_{k=1}^{m}\left(\frac{\partial g}{\partial s_k}\frac{\partial u_k}{\partial z_j} + \frac{\partial g}{\partial t_k}\frac{\partial v_k}{\partial z_j}\right) \\
&= \sum_{m=1}^{m}\left(\left[\frac{\partial g}{\partial w_k} + \frac{\partial g}{\partial \bar{w}_k}\right]\frac{\partial u_k}{\partial z_j} + \left[i\frac{\partial g}{\partial w_k} - i\frac{\partial g}{\partial \bar{w}_k}\right]\frac{\partial v_k}{\partial z_j}\right) \\
&= \sum_{m=1}^{m}\left(\frac{\partial g}{\partial w_k}\left[\frac{\partial u_k}{\partial z_j} + i\frac{\partial v_k}{\partial z_j}\right] + \frac{\partial g}{\partial \bar{w}_k}\left[\frac{\partial u_k}{\partial z_j} - i\frac{\partial v_k}{\partial z_j}\right]\right) \\
&= \sum_{m=1}^{m}\left(\frac{\partial g}{\partial w_k}\frac{\partial f_k}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_k}\frac{\partial \bar{f}_k}{\partial z_j}\right)
\end{aligned}$$

Similarly

$$\frac{\partial}{\partial \bar{z}_j}(g \circ f)(z) = \sum_{m=1}^{m}\left(\frac{\partial g}{\partial w_k}\frac{\partial f_k}{\partial \bar{z}_j} + \frac{\partial g}{\partial \bar{w}_k}\frac{\partial \bar{f}_k}{\partial \bar{z}_j}\right).$$

Now notice that if both f and g are holomorphic, then

$$\frac{\partial f_k}{\partial \bar{z}_j} = 0, \text{ and } \frac{\partial g}{\partial \bar{w}_k} = 0, \quad \text{for all } k = 1, \dots, m, \text{ and all } j = 1, \dots, n$$

hence

$$\frac{\partial}{\partial \bar{z}_j}(g \circ f)(z) = 0.$$

Therefore $g \circ f$ is holomorphic.

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Q-2) Let $\mathcal{F}$ be a sheaf on a complex manifold M and let $\mathcal{U} = \{U, V, W, Z, T, \dots\}$ be an open covering. Let $\sigma = \{\sigma_{UVW} \mid U, V, W \in \mathcal{U}\} \in C^2(\mathcal{U}, \mathcal{F})$. Let as usual $\delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ be the Čech co-boundary operator.
Explicitly write $(\delta\sigma)_{UVWZ}$.
Explicitly write $(\delta(\delta\sigma))_{UVWZT}$ and simplify it to see that it vanishes.
In the above calculations ignore the restriction maps to avoid garbage of notation.

Solution:

$$(\delta\sigma)_{UVWZ} = \sigma_{VWZ} - \sigma_{UWZ} + \sigma_{UVZ} - \sigma_{UVW}.$$

$$\begin{aligned}
((\delta\sigma))_{UVWZT} &= (\delta\sigma)_{VWZT} - (\delta\sigma)_{UWZT} + (\delta\sigma)_{UVZT} - (\delta\sigma)_{UVWT} + (\delta\sigma)_{UVWZ} \\
&= \quad \sigma_{WZT} - \sigma_{VZT} + \sigma_{VWT} - \sigma_{VWZ} \\
&\quad -\sigma_{WZT} + \sigma_{UZT} - \sigma_{UWT} + \sigma_{UWZ} \\
&\quad +\sigma_{VZT} - \sigma_{UZT} + \sigma_{UVT} - \sigma_{UVZ} \\
&\quad -\sigma_{VWT} + \sigma_{UWT} - \sigma_{UVT} + \sigma_{UVW} \\
&\quad +\sigma_{VWZ} - \sigma_{UWZ} + \sigma_{UVZ} - \sigma_{UVW} \\
&= 0.
\end{aligned}$$