Due Date: 12 May 2017, Friday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:...

STUDENT NO:.......................................

Math 430 / Math 505 Introduction to Complex Geometry – Homework 5 – Solutions

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Submit your solutions on this booklet only. Use extra pages if necessary.

Rules for Homework and Take-Home Exams

(1) You may discuss the problems with your classmates or with me, or even with people who took this course before. It is not considered good behavior to ask these questions at online forums without mentioning that these are homework questions of an introductory course.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) **It is absolutely mandatory that you write your answers alone.**

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends. *(It is always nice to flatter your friends by using their ideas and thanking them.)*

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

(6) Do not lend your written work to your friends and do not ask to borrow their written work. You may explain your solutions to your friends to any degree of detail you like, or you may ask them as many questions as they are willing to answer. **But the final writing process should be done alone.**

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work. Moreover I declare that every solution I wrote reflects my true understanding of the problem, and any sources used, including ideas from friends, are explicitly cited without exception.

Please sign here: *Date:*

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Q-1) Let D and D' be two linearly equivalent divisors on some complex manifold M. Show that the line bundles associated to these divisors are the same.

Solution:

Choose an open covering $\{U_\alpha\}$ of M such that the divisors D and D' are given by the local data $\{f_\alpha, U_\alpha\}$ and $\{f'_\alpha, U_\alpha\}$ respectively, for some meromorphic functions f_α and f'_α. This means, for example, that $D \cap U_\alpha = Z(f_\alpha)$.

The line bundle defined by D is given by the transition functions $\{g_{\alpha\beta}\}$ where

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}.$$

Since $D \sim D'$, there exists a meromorphic function h on M such that $f'_\alpha = hf_\alpha$. The transition functions for the line bundle associated to D' are then found as

$$g'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta} = \frac{hf_\alpha}{hf_\beta} = g_{\alpha\beta}.$$

Hence the two line bundles are the same.

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Q-2) Recall that a line bundle L on a complex manifold M is represented by an element ℓ in $H^1(M, \mathcal{O}^*)$. In the long exact cohomology sequence associated to the exponential sequence, this element ℓ maps into $H^2(M, \mathbb{Z})$. The image is denoted by $c_1(L)$ and is called the first Chern class of L. Show that

(i) Every line bundle on $\mathbb{P}^n$ is uniquely determined by its first Chern class.

(ii) Every divisor on $\mathbb{P}^n$ is linearly equivalent to a multiple of the hyperplane divisor.

Solution:

(i) Here we recall that

$$H^p(\mathbb{P}^n, \mathcal{O}) = 0 \quad \text{for } p > 0 \quad ,$$

see Griffiths & Harris page 49 and put $q = 0$.

It follows from the cell decomposition of the complex projective space $\mathbb{P}^n$ that

$$H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z},$$

see Griffiths & Harris page 60.

Now consider the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

and the following portion of the associated exact sequence of cohomology

$$\cdots \to H^1(\mathbb{P}^n, \mathcal{O}) \to H^1(\mathbb{P}^n, \mathcal{O}^*) \stackrel{c_1}{\to} H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathcal{O}) \to \cdots$$

Since the two groups at the ends are zero, the Chern class map c_1 in the middle is an isomorphism. This means that every line bundle on $\mathbb{P}^n$ is uniquely determined by its first Chern class.

(ii) Let $x_0, \ldots, x_n$ be the homogeneous coordinates of $\mathbb{P}^n$. Let D be a divisor given as the zero set of a homogeneous polynomial $P(x)$ of degree d. Consider the rational function

$$\phi(x) = \frac{P(x)}{x_0^d}$$

on $\mathbb{P}^n$. Note that

$$Z(x_0) = H, \quad \text{a hyperplane divisor.}$$

Now considering the zero set of ϕ we get as divisors

$$(\phi) = D - (x_0^d) = D - d(x_0) = D - dH.$$

Thus every divisor is a multiple of the hyperplane divisor up to linear equivalence.