Bilkent University

Exam # 01
Math 430 Introduction to Complex Geometry
Due: 12 March 2020
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Let $F_n = \{x+iy \in \mathbb{C} \mid -\infty < x < \infty, \quad (2n-1)\pi < y < (2n+1)\pi \}$, where $n \in \mathbb{Z}$. Also let $G = \mathbb{C} \setminus \{x+iy \in \mathbb{C} \mid y = 0, \quad x \leq 0 \}$, where $\setminus$ denotes subtraction of sets.
Define a map $\exp : \mathbb{C} \to \mathbb{C}$ by $\exp(z) = e^x \cos y + ie^x \sin y$.

(i) Show that $\exp$ is analytic.

(ii) Show that $\exp$ is the unique analytic extension to $\mathbb{C}$ of the real analytic function e^x.

(iii) Show that $\exp : F_n \to G$ is one-to-one and onto.

(iv) Define an inverse of $\exp : F_n \to G$ for each n. It is easier to describe the inverse function using polar coordinates in G.

(v) Show that each of the above inverses, which we call a branch of the log function, is analytic. It is again easier here to use the polar version of the Cauchy-Riemann equations.

Solution:

(i) Here $u = e^x \cos y$ and $v = e^x \sin y$. One checks immediately that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at every point of $\mathbb{C}$. Hence $\exp$ is analytic everywhere.

(ii) Let $g(z)$ be another analytic extension of the real analytic e^x. Then $\exp z$ and $g(z)$ agree at every point of the real line, i.e. the entire function $\exp z - g(z)$ has a zero set with accumulation points, and hence is identically zero. This shows that there can be only one extension to $\mathbb{C}$ of any real analytic function.

(iii) For notational convenience let $f(z) = \exp z$, $z \in F_n$.

If $f(z_1) = f(z_2)$ for some $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ in F_n, then in particular $|f(z_1)| = |f(z_2)|$ which gives $e^{x_1} = e^{x_2}$. Since the real exponential function is one-to-one, we get $x_1 = x_2$. We then have $\cos y_1 + i \sin y_1 = \cos y_2 + i \sin y_2$. Forcing $(2n-1)\pi < y_1, y_2 < (2n+1)\pi$ gives us $y_1 = y_2$. Hence the injectivity of f on F_n.

For the surjectivity, let $\alpha \in G$. Then in polar coordinates we can write $\alpha = re^{i\theta}$ where $-\pi < \theta < \pi$ and $r = |\alpha| > 0$. Then $\exp((\ln r) + i(\theta + 2n\pi)) = \alpha$, giving us the surjectivity.

(iv) For $\alpha = re^{i\theta}$ in polar coordinates in G, define $\log \alpha = \ln r + i(\theta + 2n\pi)$. This is the required inverse of $\exp$ on F_n.

(v) Here we use the polar form of the Cauchy-Riemann equations: $u_r = \dfrac{1}{r}\, v_\theta$, $v_r = -\frac{1}{r}\, u_\theta$. Here $u = \ln r$ and $v = \theta + 2n\pi$. And the Cauchy-Riemann equations are clearly satisfied everywhere on the domain.

Q-2) Let $f(w) = f(s,t)$ where $w = s + it$, and $g(z) = g(x,y)$ where $z = x + iy$ be two C^∞ functions of the real variables s, t and x, y respectively. Assume that $\phi(z) = (f \circ g)(z)$ is defined. Show that

$$\frac{\partial \phi}{\partial z} = \frac{\partial f}{\partial w}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{g}}{\partial z},$$

and

$$\frac{\partial \phi}{\partial \bar{z}} = \frac{\partial f}{\partial w}\frac{\partial g}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{g}}{\partial \bar{z}},$$

where the derivatives of f are evaluated at $w = g(z)$.

Solution:

This is a straightforward tedious calculation. I will show how to do the calculations only for the first part. The second part is done in an identical manner.

Let $f(w) = u(s,t) + iv(s,t)$, where $w = s + it$, and let $g(z) = \alpha(x,y) + i\beta(x,y)$ where $z = x + iy$.

We define $\phi(z) = (f \circ g)(z) = u(\alpha(x,y), \beta(x,y)) + iv(\alpha(x,y), \beta(x,y))$.

Then $\phi_x = u_s\alpha_x + u_t\beta_x + iv_s\alpha_x + iv_t\beta_x$, and similarly $\phi_y = u_s\alpha_y + u_t\beta_y + iv_s\alpha_y + iv_t\beta_y$.

By definition we have

$$\begin{aligned}
\phi_z &= \frac{1}{2}(\phi_x - i\phi_y) \\
&= \frac{1}{2}(u_s\alpha_x + u_t\beta_x + v_s\alpha_y + v_t\beta_y) + \frac{i}{2}(v_s\alpha_x + v_t\beta_x - u_s\alpha_y - u_t\beta_y).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
f_w g_z + f_{\bar{w}}\bar{g}_z &= \frac{1}{4}(f_s - if_t)(g_x - ig_y) + \frac{1}{4}(f_s + if_t)(\bar{g}_x - i\bar{g}_y) \\
&= \frac{1}{4}(u_s + iv_s - iu_t + v_t)(\alpha_x + i\beta_x) - i\alpha_y + \beta_y) \\
&\quad + \frac{1}{4}(u_s + iv_s) + iu_t - v_t)(\alpha_x - i\beta_x - i\alpha_y - \beta_y) \\
&= \frac{1}{2}(u_s\alpha_x + u_t\beta_x + v_s\alpha_y + v_t\beta_y) + \frac{i}{2}(v_s\alpha_x + v_t\beta_x - u_s\alpha_y - u_t\beta_y).
\end{aligned}$$

This gives the required equality. The other part is similar.

Q-3) Let $f(w_1, \ldots, w_n)$ be a C^∞ function of the real variables $s_1, t_1, \ldots, s_n, t_n$ where each $w_k = s_k + it_k$. Assume further that each $w_k = g_k(z_1, \ldots, z_m)$ is a C^∞ function of the real variables $x_1, y_1, \ldots, x_m, y_m$, where each $z_j = x_j + iy_j$.
Using the previous result, give a convincing argument that for each $j = 1, \ldots, m$,

$$\frac{\partial f \circ g}{\partial z_j} = \sum_{k=1}^{n} \frac{\partial f}{\partial w_k}\frac{\partial g_k}{\partial z_j} + \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{w}_k}\frac{\partial \overline{g}_k}{\partial z_j},$$

and

$$\frac{\partial f \circ g}{\partial \overline{z}_j} = \sum_{k=1}^{n} \frac{\partial f}{\partial w_k}\frac{\partial g_k}{\partial \overline{z}_j} + \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{w}_k}\frac{\partial \overline{g}_k}{\partial \overline{z}_j},$$

where each derivative of f is evaluated at $(g_1(z), \ldots, g_n(z))$.

Solution:

This is elementary but extremely tedious. The crucial part of the solution is to get the notation right. First observe that we can take $m = 1$ since the required equalities are calculated for single z_j while the other z are kept constant. Also we can take $n = 2$ since once we know how to pass from one variable to two variables, the rest will just be a matter of notation.

Then we can write:

(I) $f(w_1, w_2) = u(s_1, t_1, s_2, t_2) + iv(s_1, t_1, s_2, t_2)$.

(II) $g_k(z) = \alpha_k(x, y) + i\beta_k(x, y)$, $k = 1, 2$.

(III) $\phi(z) = f(g_1(z), g_2(z)) = u(\alpha_1(x, y), \beta_1(x, y), \alpha_2(x, y), \beta_2(x, y))$

Since each of ϕ_z, f_{w_k} and $g_z ermsof$ are written in tthe real variables, the problem reduces to calculating the both sides of the claim using the real Calculus chain rule and simplifying as in the second question.