Bilkent University

Exam # 03
Math 430, Math 505 Introduction to Complex Geometry
Due: 8 May 2020
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Solution Key

Q-1) *We start by reminding numerous notations. The questions follow the descriptions at the end.*

Let V be an n-dimensional real inner product space with $\langle \cdot, \cdot \rangle$ denoting the inner product. Let $\Lambda(V) = \oplus_{k=0}^{n} \Lambda_k(V)$ be the exterior algebra on V. We extend the inner product of V to $\Lambda(V)$ as follows. If $u \in \Lambda_r(V)$ and $v \in \Lambda_s(V)$, then $\langle u, v \rangle = 0$ if $r \neq s$. When $r = s$, let $u = u_1 \wedge \cdots \wedge u_r$, $v = v_1 \wedge \cdots \wedge v_r$, where $u_i, v_j \in V$, then we set

$$\langle u_1 \wedge \cdots \wedge u_r, v_1 \wedge \cdots \wedge v_r \rangle = \det(\langle u_i, v_j \rangle),$$

and extend this to $\Lambda_r(V)$ linearly.

Let $e_1, \ldots, e_n$ be a basis of V. For any subset $I = \{i_1, \ldots, i_d\} \subseteq \{1, \ldots, n\}$, define

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_d},$$

and define $e_\emptyset = 1$, where $\emptyset$ is the emptyset.

Since $\Lambda_n(V)$ is a one dimensional real space, $\Lambda_n(V) - \{0\}$ has two components. An orientation of V is a choice of one of these components. V is oriented if one such choice is made. If $e_1, \ldots, e_n$ is a basis of V, we say that $e_1, \ldots, e_n$ is positively oriented if $e_1 \wedge \cdots \wedge e_n$ is in the chosen component of $\Lambda_n(V) - \{0\}$.

If V is an oriented real inner product space, there is a linear map

$$* : \Lambda(V) \to \Lambda(V),$$

called the star map which is defined as follows. Let $e_1, \ldots, e_n$ be an orthonormal basis of V, not necessarily positively oriented. Then we set

$$*(1) = \pm e_1 \wedge \cdots \wedge e_n, \quad *(e_1 \wedge \cdots \wedge e_n) = \pm 1,$$
$$*(e_1 \wedge \cdots \wedge e_p) = \pm e_{p+1} \wedge \cdots \wedge e_n,$$

where one takes "+" if $e_1 \wedge \cdots \wedge e_n$ is positively oriented, and "$-$" otherwise. We then extend this definition linearly to all of $\Lambda(V)$.

Moreover for any $\alpha \in V$, let L_α be the left multiplication by α in the algebra $\Lambda(V)$, i.e. for any $\gamma \in \Lambda(V)$, we define $L_\alpha(\gamma) = \alpha \wedge \gamma$. Let L_α^* be its adjoint, i.e. for any $\beta \in \Lambda_p(V)$ and $\gamma \in \Lambda_{p+1}(V)$, we have $\langle L_\alpha(\beta), \gamma \rangle = \langle \beta, L_\alpha^*(\gamma) \rangle$.

Our exam questions now follow.

We assume throughout that V is an oriented real inner product space of dimension n.

(i) Let $e_1, \dots, e_n$ be an orthonormal basis of V. Show that the collection

$$\{e_I \mid I = \{i_1, \dots, i_d\} \text{ is a subset of } \{1, \dots, n\} \quad \text{with } \ i_1 < \cdots < i_d, \ \text{ for } \ d = 0, \dots, n\}$$

is an orthonormal basis of $\Lambda(V)$. Here when $d = 0$, we take I as the empty set $\emptyset$, and assign $e_\emptyset = 1$.

(ii) Prove that for $\alpha \in \Lambda_p(V)$, we have

$$**(\alpha) = (-1)^{p(n-p)}\alpha.$$

(iii) Prove that for $\alpha, \beta \in \Lambda_p(V)$, we have

$$\langle \alpha, \beta \rangle = *(\alpha \wedge *\beta) = *(\beta \wedge *\alpha).$$

(iv) Show that for any $\gamma \in \Lambda_{p+1}(V)$, we have

$$L_\alpha^*(\gamma) = (-1)^{np} * L_\alpha(*\gamma).$$

Remark: These are composed from Exercises 13 and 14 on pages 79-80 of Frank W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, 1983. You may find it helpful to read pages 54-57 of this book for the above problems. Also notice that all the operations are extended linearly, so proving certain identities only on nice basis elements may suffice for the general case if you argue convincingly.

Solution (i) That this collection spans $\Lambda(V)$ is clear. So we show orthonormality. From the definition of the inner product on $\Lambda(V)$, we immediately see that $\langle e_I, e_J \rangle = 0$ if $\#I \neq \#J$. Now suppose $\#I = \#J = d > 0$ but $I \neq J$. Then there is an $i \in I$ such that $i \notin J = \{j_1, \dots, j_d\}$. Then $\langle e_i, e_{j_k} \rangle = 0$ for $k = 1, \dots, d$. Hence the i-th row of the matrix $(\langle e_{i_s}, e_{j_t} \rangle)_{1 \le s,t \le d}$ will be zero, where we set $I = \{i_1, \dots, i_d\}$. Hence by definition $\langle e_I, e_J \rangle = 0$. Whereas clearly $\langle e_I, e_I \rangle = 1$ by definition. Hence the given collection is an othonormal basis.

Solution (ii) It surely suffices to prove this for the orthonormal basis given in part (i). For this purpose let $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_p}$ with $1 \le i_1 < \cdots < i_p \le n$. Let $\{j_1, \dots, j_{n-p}\}$ be the complement of $\{i_1, \dots, i_p\}$ in $\{1, \dots, n\}$, where again $1 \le j_1 < \cdots < j_{n-p} \le n$. Let $\epsilon \in \{-1, +1\}$ be such that

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{n-p}} = \epsilon\, e_1 \wedge \cdots \wedge e_n.$$

This means that

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \epsilon\, e_{j_1} \wedge \cdots \wedge e_{j_{n-p}}.$$

On the other hand, it is trivial to check that

$$\begin{aligned} e_{j_1} \wedge \cdots \wedge e_{j_{n-p}} \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} &= (-1)^{p(n-p)} e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{n-p}} \\ &= \epsilon\, (-1)^{p(n-p)} e_1 \wedge \cdots \wedge e_n, \end{aligned}$$

which says that

$$*(e_{j_1} \wedge \cdots \wedge e_{j_{n-p}}) = \epsilon\, (-1)^{p(n-p)} e_{i_1} \wedge \cdots \wedge e_{i_p}.$$

Now we have

$$\begin{aligned}
*(e_{i_1} \wedge \cdots \wedge e_{i_p}) &= \epsilon\, e_{j_1} \wedge \cdots \wedge e_{j_{n-p}} \\
** (e_{i_1} \wedge \cdots \wedge e_{i_p}) &= \epsilon\ * (e_{j_1} \wedge \cdots \wedge e_{j_{n-p}}) \\
&= \epsilon^2 (-1)^{p(n-p)} e_{i_1} \wedge \cdots \wedge e_{i_p} \\
&= (-1)^{p(n-p)} e_{i_1} \wedge \cdots \wedge e_{i_p},
\end{aligned}$$

as claimed.

Solution (iii) Again by linearity, it suffices to prove this when α, β are elements of the orthogonal basis given in (i).

If I, J are subsets of $\{1, \ldots, n\}$ and $I \neq J$, then without loss of generality we may assume that there is $i \in I$ such that $i \notin J$. Then e_i is a component of both e_I and $*e_J$, hence $e_I \wedge *e_J = 0$. Also from (i) we know that $\langle e_I, e_J \rangle = 0$. Therefore it remains to prove the required equality when $I = J$. In this case we first have $\langle e_I, e_I \rangle = 1$. Now using the notation of (ii) above, we have

$$\begin{aligned}
*(e_{i_1} \wedge \cdots \wedge e_{i_p}) &= \epsilon\, e_{j_1} \wedge \cdots \wedge e_{j_{n-p}} \\
e_I \wedge *e_I &= \epsilon\, e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{n-p}} \\
&= \epsilon^2\, e_1 \wedge \cdots \wedge e_n, \\
&= e_1 \wedge \cdots \wedge e_n. \\
*(e_I \wedge *e_I) &= 1.
\end{aligned}$$

Hence the required equality holds.

Solution (iv) We will show that

$$\langle \theta, L_\alpha^*(\gamma) \rangle = \langle \theta, (-1)^{np} * L_\alpha(*\gamma) \rangle, \quad \text{for any } \theta \in \Lambda_p(V). \tag{A}$$

For this we will use the identities we proved in (iii). Recall that $\alpha \in \Lambda_1(V) = V$, and $L_\alpha(\gamma) = \alpha \wedge \gamma$.

We first have

$$\begin{aligned}
\langle \theta, L_\alpha^*(\gamma) \rangle &= \langle L_\alpha(\theta), \gamma \rangle = \langle \alpha \wedge \theta, \gamma \rangle \\
&= *(\alpha \wedge \theta \wedge *\gamma) \\
&= (-1)^p * (\theta \wedge \alpha \wedge *\gamma).
\end{aligned} \tag{B}$$

Next we have

$$\begin{aligned}
\langle \theta, (-1)^{np} * L_\alpha(*\gamma) \rangle &= \langle \theta, (-1)^{np} * (\alpha \wedge *\gamma) \rangle \\
&= (-1)^{np} * (\theta \wedge ** (\alpha \wedge *\gamma)) \\
&= (-1)^{np} (-1)^{p(n-p)} * (\theta \wedge \alpha \wedge *\gamma) \\
&= (-1)^p * (\theta \wedge \alpha \wedge *\gamma),
\end{aligned} \tag{C}$$

since $np + p(n-p) \equiv p \mod 2$.

Since (B) and (C) agree, (A) holds. This completes the proof.