Exam # 03
Math 430, Math 505 Introduction to Complex Geometry
Due: 8 May 2020
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Q-1) *We start by reminding numerous notations. The questions follow the descriptions at the end.*

Let V be an n-dimensional real inner product space with $\langle \cdot, \cdot \rangle$ denoting the inner product. Let $\Lambda(V) = \oplus_{k=0}^{n} \Lambda_k(V)$ be the exterior algebra on V. We extend the inner product of V to $\Lambda(V)$ as follows. If $u \in \Lambda_r(V)$ and $v \in \Lambda_s(V)$, then $\langle u, v \rangle = 0$ if $r \neq s$. When $r = s$, let $u = u_1 \wedge \cdots \wedge u_r$, $v = v_1 \wedge \cdots \wedge v_r$, where $u_i, v_j \in V$, then we set

$$\langle u_1 \wedge \cdots \wedge u_r, v_1 \wedge \cdots \wedge v_r \rangle = \det(\langle u_i, v_j \rangle),$$

and extend this to $\Lambda_r(V)$ linearly.

Let $e_1, \ldots, e_n$ be a basis of V. For any subset $I = \{i_1, \ldots, i_d\} \subseteq \{1, \ldots, n\}$, define

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_d},$$

and define $e_\emptyset = 1$, where $\emptyset$ is the emptyset.

Since $\Lambda_n(V)$ is a one dimensional real space, $\Lambda_n(V) - \{0\}$ has two components. An orientation of V is a choice of one of these components. V is oriented if one such choice is made. If $e_1, \ldots, e_n$ is a basis of V, we say that $e_1, \ldots, e_n$ is positively oriented if $e_1 \wedge \cdots \wedge e_n$ is in the chosen component of $\Lambda_n(V) - \{0\}$.

If V is an oriented real inner product space, there is a linear map

$$* : \Lambda(V) \to \Lambda(V),$$

called the star map which is defined as follows. Let $e_1, \ldots, e_n$ be an orthonormal basis of V, not necessarily positively oriented. Then we set

$$*(1) = \pm e_1 \wedge \cdots \wedge e_n, \quad *(e_1 \wedge \cdots \wedge e_n) = \pm 1,$$
$$*(e_1 \wedge \cdots \wedge e_p) = \pm e_{p+1} \wedge \cdots \wedge e_n,$$

where one takes "+" if $e_1 \wedge \cdots \wedge e_n$ is positively oriented, and "$-$" otherwise. We then extend this definition linearly to all of $\Lambda(V)$.

Moreover for any $\alpha \in V$, let L_α be the left multiplication by α in the algebra $\Lambda(V)$, i.e. for any $\gamma \in \Lambda(V)$, we define $L_\alpha(\gamma) = \alpha \wedge \gamma$. Let L_α^* be its adjoint, i.e. for any $\beta \in \Lambda_p(V)$ and $\gamma \in \Lambda_{p+1}(V)$, we have $\langle L_\alpha(\beta), \gamma \rangle = \langle \beta, L_\alpha^*(\gamma) \rangle$.

Our exam questions now follow.
We assume throughout that V is an oriented real inner product space of dimension n.

(i) Let $e_1, \ldots, e_n$ be an orthonormal basis of V. Show that the collection

$$\{e_I \mid I = \{i_1, \ldots, i_d\} \text{ is a subset of } \{1, \ldots, n\} \quad \text{with } \; i_1 < \cdots < i_d, \; \text{ for } \; d = 0, \ldots, n\}$$

is an orthonormal basis of $\Lambda(V)$. Here when $d = 0$, we take I as the empty set $\emptyset$, and assign $e_\emptyset = 1$.

(ii) Prove that for $\alpha \in \Lambda_p(V)$, we have

$$* * (\alpha) = (-1)^{p(n-p)} \alpha.$$

(iii) Prove that for $\alpha, \beta \in \Lambda_p(V)$, we have

$$\langle \alpha, \beta \rangle = *(\alpha \wedge * \beta) = *(\beta \wedge * \alpha).$$

(iv) Show that for any $\gamma \in \Lambda_{p+1}(V)$, we have

$$L_\alpha^*(\gamma) = (-1)^{np} * L_\alpha(*\gamma).$$

Remark: These are composed from Exercises 13 and 14 on pages 79-80 of
Frank W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, 1983.
You may find it helpful to read pages 54-57 of this book for the above problems. Also notice that all the operations are extended linearly, so proving certain identities only on nice basis elements may suffice for the general case if you argue convincingly.