Math 431 ALGEBRAIC GEOMETRY
Homework 2 Solution Key

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1) Exercise 8.3 page 93: Show in particular that s is a ramification point of $f(x)$ of multiplicity k if and only if s is a root of $f'(x)$ of multiplicity $k-1$.

Assume that s is a ramification point of f with index k. Then $f(x) - f(s) = (x-s)^k g(x)$ with $g(s) \neq 0$. Now $f'(x) = (x-s)^{k-1}h(x)$ where $h(x) = kg(x) + (x-s)g'(x)$. Note that $h(s) = sg(s) \neq 0$, so s is a root of $f'(x)$ with multiplicity $k-1$.

Conversely assume that s is a root of $f'(x)$ with multiplicity $k-1$. Let $f(x)-f(s) = (x-s)^t g(x)$ for some integer $t \geq 0$ and some polynomial $g(x)$ with $g(s) \neq 0$. Then $f'(x) = (x-s)^{t-1}h(x)$, where $h(x) = tg(x) + (x-s)g'(x)$. Note that $h(s) = tg(s) \neq 0$. This gives s as a root of $f'(x)$ with multiplicity $t-1$, so $t = k$ and s is a ramification point of $f(x)$ with index k.

Another solution for this second part, which was popular on the homework papers is the following: Let $f'(x) = (x-s)^{k-1}h(x)$ with $h(s) \neq 0$. Let the degree of h be m. $f(x) - f(s) = \int_s^x (z-s)^{k-1}h(z)dz$. Using integration by parts m times we get $f(x) - f(s) = \frac{1}{k}(x-s)^k h(x) - \frac{1}{k}\frac{1}{k+1}(x-s)^{k+1}h''(x) + \cdots \pm \frac{1}{k}\frac{1}{k+1}\cdots\frac{1}{k+m-1}(x-s)^{k+m}h^{(m)}(x)$. From here it follows immediately that s is a ramification point of $f(x)$ with index k.

To check the answer with the Riemann-Hurwitz formula let R be the ramification divisor of f where we consider f as a holomorphic mapping from $\mathbb{P}^1$ to $\mathbb{P}^1$. Clearly ∞ is a ramification point with index $n-1$ where $n = \deg f$. Assume that $R = (n-1)\infty + \sum_{i=1}^r m_i p_i$. The above argument shows that $f'(x) = (x-p_1)^{m_1}\cdots(x-p_r)^{m_r}$. We now have $m_1 + \cdots + m_r = \deg f' = n-1$. Thus we find the degree of the ramification divisor as $2(n-1)$. On the other hand the Riemann-Hurwitz formula gives $\deg R = 2(g + n - g'n - 1)$, which gives $2(n-1)$ after substituting $g = g' = 0$.

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2) Exercise 7.5 page 89: If an n th degree curve has $[\frac{n}{2}] + 1$ singular points on a straight line L, then L is necessarily a curve component of this curve.

By Bezout's theorem $\displaystyle\sum_{p\in C\cap L}(L\cdot C)_p = \deg L \cdot \deg C = n$. On the other hand $\displaystyle\sum_{p\in C\cap L}(L\cdot C)_p =$
$$\sum_{p\in C\cap L,\ p \text{ singular}}(L\cdot C)_p + \sum_{p\in C\cap L,\ p \text{ smooth}}(L\cdot C)_p \geq \sum_{p\in C\cap L,\ p \text{ singular}}(L\cdot C)_p \geq 2([\frac{n}{2}]+1) > n,$$
since each $(L\cdot C)_p \geq 2$ when p is singular on C. But this contradicts Bezout's theorem. So L must be a component of C. For the proof of $(L\cdot C)_p \geq 2$ when p is singular, see either the definition 7.3 on page 83, or see the hint to exercise 7.3 on page 85.

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3) Show that every smooth algebraic plane curve C is irreducible.

Let C be the zero set of the polynomial f. Suppose C is not irreducible. Then $f = gh$ for some nontrivial polynomials g and h. The curves $V(g)$ and $V(h)$ intersect at a point p in $\mathbb{P}^2$. Let x and y be the affine coordinates at p. Then we have $f(x,y) = g(x,y)h(x,y)$ and $\frac{\partial f}{\partial x}(p) = \frac{\partial g}{\partial x}(p)h(p) + \frac{\partial h}{\partial x}(p)g(p) = 0$ since p is both on $V(g)$ and $V(h)$. Similarly $\frac{\partial f}{\partial y}(p) = 0$. But this means that p is a singular point of C contradicting the fact that C is smooth. So C must be irreducible.
