Math 431 ALGEBRAIC GEOMETRY
Homework 3 Solution Key

Ali Sinan Sertöz 7 January 2003

---

1) Show that for any $g \geq 0$, there is a compact Riemann surface of genus g. In particular show that for any $g > 2$ there exist a hyperelliptic and a non-hyperelliptic compact Riemann surface of genus g. You may not be able to give the proofs in detail but you can quote some technical results with full references which imply the existence of the required surfaces.

For $g = 0$ and $g = 1$ we have the projective line and the torus respectively.

To construct a hyperelliptic curve of genus $g \geq 2$ we quote your text book page 141; for any given $2g + 2$ mutually distinct points $a_1, \ldots, a_{2g+2}$ in $\mathbb{C}$ we first consider the curve

$$C = \left\{ y^2 - \prod_{i=1}^{2g+2} (x - a_i) = 0 \right\} \cup [0 : 0 : 1] \subset \mathbb{P}^1.$$

The normalization C' of this curve is a hyperelliptic curve of genus g. Now the remark on page 144 of your textbook says that two such hyperelliptic curves are isomorphic if and only if the associated points $a_1, \ldots, a_{2g+2}$ are in the same orbit of the action of the automorphisms of $\mathbb{P}^1$. Since we can always send a_1, a_2, a_3 to $0, 1, \infty$ under such an isomorphism, the moduli of hyperelliptic curves of genus g is of dimension $2g - 1$.

Now we quote from Griffiths & Harris, *Principles of Algebraic Geometry*, page 256, where it is proved that the general Riemann surface of genus g depends on $3g - 3$ parameters.

It follows that when $g > 2$, the collection of hyperelliptic curves is a proper subset of the collection of Riemann surfaces and hence there exists, in abundance, nonhyperelliptic curves of genus $g \geq 3$.

An alternate way of finding nonhyperelliptic curves is to use Remark 5.5.1 on page 345 of Hartshorne, *Algebraic Geometry*. Here we denote with g^r_d any complete linear system $|D|$ where $\deg D = d$ and $\dim |D| = r$. Then this g^r_d defines, using the bases elements of $\mathcal{L}(D)$ as usual, a map into $\mathbb{P}^r$. Clearly the curve is hyperelliptic if it has a g^1_2, and is nonhyperelliptic if it has no g^1_2. In this remark it is mentioned that for every d with $d < g/2 + 1$, there exists a curve of genus g which contains no g^1_d. When $g \geq 3$, this amounts to saying that for every g there exist curves of genus g containing no g^1_2. Such curves are by definition nonhyperelliptic.
