Date: 2 November 2002 Saturday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

Math 431 ALGEBRAIC GEOMETRY
Midterm Exam
Solution Key

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1) Find necessary and sufficient conditions on a and b, where $a, b \in \mathbb{C}$, so that the the equation

$$f(x,y) = y^2 - 4x^3 - ax - b = 0$$

represents a smooth curve in $\mathbb{C}^2$.

This is Exercise 9.1 on page 37.
The curve, call it C, is singular at the point (x, y) if and only if the system of equations

$$\begin{aligned} f(x,y) &= y^2 - 4x^3 - ax - b = 0 \\ f_x(x,y) &= -12x^2 - a = 0 \\ f_y(x,y) &= 2y = 0 \end{aligned}$$

has a solution. The last equation forces $y = 0$ and the existence of a simultaneous solution for the first two equations is equivalent for the polynomial $g(x) = 4x^3 + ax + b$ to have a multiple root. This is the case when $\mathcal{D}(g) = \mathcal{R}(g, g') = -16(a^3 + 27b^2) = 0$, see Corollary 2.1 on page 59. Then a necessary and sufficient condition for the curve C to be smooth is $a^3 + 27b^2 \neq 0$.

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2) Let $\mathbb{C}/\Lambda$ be a torus, and let $C_{a\,b}$ denote the curve in $\mathbb{C}^2$ given by $y^2 = 4x^3 + ax + b$ where $a, b \in \mathbb{C}$. Show that $(\mathbb{C}/\Lambda)\backslash\{[0]\}$ is isomorphic to a curve $C_{a\,b}$ for a suitable choice of constants a and b, where $[0]$ denotes the equivalence class of $0 \in \mathbb{C}$.

This is Exercise 10.3 and Example 2 on page 49.
Weierstrass $\wp$ function and its derivative satisfy an equation of the form $\wp'^2 = 4\wp^3 + a\wp + b$ for some $a, b \in \mathbb{C}$, where the constants a and b depend on the lattice Λ. There is a map

$$\begin{aligned} f : \mathbb{C}/\Lambda &\longrightarrow \mathbb{P}^2 \\ [z] &\longmapsto [\wp(z) : \wp'(z) : 1] \end{aligned}$$

with the understanding that $[0]$ maps to a point at infinity, $[0 : 1 : 0]$. The equation between $\wp$ and $\wp'$ ensures that the image of this map outside $[0]$ is the curve $C_{a\,b}$.

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3) Show that any two plane algebraic curves in $\mathbb{P}^2$ intersect. Use this to show that $\mathbb{P}^1 \times \mathbb{P}^1$ cannot be isomorphic to $\mathbb{P}^2$.

This requires Theorem 2.2 on page 58.
Let $F(X, Y, Z)$ and $G(X, Y, Z)$ be the homogeneous polynomials giving the two algebraic plane curves. By a linear change of variables of the form $X \mapsto X$, $Y \mapsto Y + \lambda X$ and $Z \mapsto Z + \lambda X$, and choosing λ suitably, we can assume that F and G are written as

$$\begin{aligned} F(X,Y,Z) &= X^n + a_1 X^{n-1} + \cdots + a_n \\ G(X,Y,Z) &= X^m + b_1 X^{m-1} + \cdots + b_m \end{aligned}$$

where $a_i = a_i(Y, Z)$ and $b_i = b_i(Y, Z)$ are homogeneous of degree i, if not zero. Considering F and G as polynomials in X, let R denote their Sylvester matrix; i.e. $\det R = \mathcal{R}(F, G)(Y, Z)$ is their resultant. We are assuming that F and G have no common components, so $\det R \not\equiv 0$. To fix our notation, we write R as

$$R = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}}_{m} \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ b_1 & 1 & \cdots & 0 \\ b_2 & b_1 & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & b_m \end{pmatrix}}_{n}$$

If we adopt the convention that

$$\begin{aligned} a_t &\equiv 0 \text{ if } t < 0 \text{ or } t > n, \\ b_t &\equiv 0 \text{ if } t < 0 \text{ or } t > m, \\ a_0 &= 1 \text{ and} \\ b_0 &= 1, \end{aligned}$$

then we can easily describe $R = (R_{ij})$ as

$$R_{ij} = \begin{cases} a_{i-j} & \text{if } 1 \le j \le m, \\ b_{i+m-j} & \text{if } m+1 \le j \le m+n. \end{cases}$$

Each nonzero term in the expansion of $\det R$ is of the form

$$R_\sigma = R_{1\,\sigma(1)} R_{2\,\sigma(2)} \cdots R_{m+n\,\sigma(m+n)},$$

where σ is a permutation on $\{1, 2, \ldots, m+n\}$. We know that each $R_{i\,\sigma(i)}$ is either $a_{i-\sigma(i)}$ or $b_{i+m-\sigma(i)}$, if not zero. Therefore we can write the degree of R_σ as

$$\begin{aligned} \deg R_\sigma &= \sum_{\sigma(i) \le m} (i - \sigma(i)) + \sum_{\sigma(i) > m} (i + m - \sigma(i)) \\ &= \sum_{i=1}^{m+n} i - \sum_{i=1}^{m+n} \sigma(i) + \sum_{i=m+1}^{m+n} m. \\ &= nm. \end{aligned}$$

We see that each term is homogeneous of the same degree. So the resultant is a homogeneous polynomial in Y and Z of degree mn. For any nonzero $Z_0 \in \mathbb{C}$, the polynomial $\mathcal{R}(F, G)(Y, Z_0)$ has a root, say Y_0. This means that the polynomials $F(X, Y_0, Z_0)$ and $G(X, Y_0, Z_0)$ have a common zero, say X_0. Then the two curves intersect at $[X_0 : Y_0 : Z_0] \in \mathbb{P}^2$.

On the other hand the lines $[1:0] \times \mathbb{P}^1$ and $[0:1] \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ clearly do not intersect, hence $\mathbb{P}^1 \times \mathbb{P}^1$ cannot be isomorphic to $\mathbb{P}^2$.

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4) Show that $y^2 + xy + x^n$, $n > 2$, is irreducible in $\mathbb{C}[x, y]$ but reducible in $\mathbb{C}\{x\}[y]$.

This is about Corollary 4.6 and its proof on pages 72-73, and Exercise 4.1 on page 75.
Using the quadratic formula

$$y^2 + xy + x^n = \big(y - \frac{x}{2}(1 + \sqrt{1 - 4x^{n-2}})\big)\big(y - \frac{x}{2}(1 - \sqrt{1 - 4x^{n-2}})\big).$$

The function $\sqrt{1 - 4x^{n-2}}$ defines a holomorphic function since $n > 2$ and hence has a power series expansion and is in $\mathbb{C}\{x\}$. Thus the given polynomial y^2+xy+x^n is reducible in $\mathbb{C}\{x\}[y]$. This ring is a UFD and hence this is the only factorization of this polynomial here. If the polynomial is reducible in $\mathbb{C}[x, y]$, then $y^2+xy+x^n = f(x, y)g(x, y)$ with both f and g polynomials. But then this would also be the factorization in $\mathbb{C}\{x\}[y]$ where we know the factorization has no polynomial parts. Hence the polynomial is irreducible in the ring $\mathbb{C}[x, y]$.

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5) Draw the tangent lines at the origin, in $\mathbb{R}^2$, for the curves
  a) $x^2y + xy^2 - x^4 - y^4$.
  b) $x^2 - x^4 - y^4$.

This requires the information on page 54.
The tangent lines of the first curve at the origin are given by the equation $x^2y + xy^2 = 0$, and the second one by $x^2 = 0$, both are easy to draw.
