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Math 431 ALGEBRAIC GEOMETRY
Midterm Exam II
Solutions

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1) Show that every compact Riemann surface of genus 2 is hyperelliptic.

This is proposition 5.1 on page 144, and it uses proposition 3.5 on page 134. Here is another proof: Since $\ell(K) = g = 2$, there is a nonconstant function f such that $D = (f) + K \geq 0$. Clearly $D \sim K$ and $\deg D = \deg K = 2g - 2 = 2$, so we might as well take $K = p + q$. Let f be a nonconstant function in $\mathcal{L}(K)$. If f has a pole only at p or only at q, then f gives an isomorphism between our Riemann surface and $\mathbb{P}^1$, and the genus becomes zero. Hence f has a pole at p and q, and is the required degree two map onto $\mathbb{P}^1$. (Recall the definition of hyperelliptic on page 135.)

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2) For a positive integer $n \geq 2$ let X_n be a compact Riemann surface with the property that there is a point $p \in X_n$ such that $\ell(p) = n$. For each such n, describe the corresponding X_n.

The given condition means that there is a nonconstant function f in $\mathcal{L}(p)$. Then f is forced to have a single pole at p and be regular elsewhere. This f gives an isomorphism between X_n and $\mathbb{P}^1$, and hence the genus is zero. When genus is zero, $\deg(K - p) < 0$, so $\ell(K - p) = 0$. The Riemann-Roch theorem then gives $\ell(p) = 2$. We conclude that $X_2 \cong \mathbb{P}^1$, and X_n does not exist for $n > 2$.

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3) Let X be a compact Riemann surface of genus g, and let D be a divisor of degree 0. Show that if $\ell(D) > 0$, then $D \sim 0$. Using this or otherwise show that if E is a divisor of degree $2g - 2$ with $\ell(E) = g$, then $E \sim K$, where K denotes a canonical divisor on X.

For the first part, if $D = 0$ then there is nothing further to prove. If $D \neq 0$ then D has some positive and some negative parts, so $\mathcal{L}(D)$ does not contain constants. Since $\ell(D) > 0$, there is a nonconstant function f such that $F = (f) + D \geq 0$. F is an effective divisor linearly equivalent to D and has degree 0. So $F = 0$ and the result follows. For the second part let $D = K - E$ and apply the first part; if $E = K$, then we are done. If not, then D is a nonzero divisor of degree zero, so $\mathcal{L}(D)$ does not contain the constants. To see if it contains any other function we apply Rieamnn-Roch formula for $D = K - E$. Recalling that $\ell(K - (K - E)) = \ell(E) = g$, we find from $\ell(K - E) - \ell(K - (K - E)) = \deg(K - E) - g + 1$ that $\ell(K - E) = 1$. Now the first part tells us that $K - E \sim 0$, or equivalentlt $E \sim K$.

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4) Let D be a divisor on a compact Riemann surface X with $\ell(D) = d+1 > 0$, and let $f_0, \ldots, f_d$ be a basis of $\mathfrak{L}(D)$. Define $\phi_D(p) = [f_0(p) : \cdots : f_d(p)]$ for $p \in X$. Show that ϕ_D defines a map into $\mathbb{P}^d$ if $\ell(D-p) = d$ for every $p \in X$. Using this or otherwise show that ϕ_K defines a map from X into $\mathbb{P}^{g-1}$, where K is a canonical divisor and $g > 1$ is the genus of X.

ϕ_D defines map into projective space if there is no point $p \in X$ with $f_0(p) = \cdots = f_d(p) = 0$. If all f_i's vanish at a point p, then every f in $\mathfrak{L}(D)$ also vanish at p since f_i's form a basis for this vector space. This however implies that $\mathfrak{L}(D-p) = \mathfrak{L}(D)$. Since $\mathfrak{L}(D-p) = d$, these two vector spaces cannot be the same and ϕ_D defines a map into projective space. For the second part use the Riemann-Roch formula to calculate $\ell(K-p)$; $\ell(K-p) - \ell(K-(K-p)) = \deg(K-p) - g + 1$. Here note that $\ell(K-(K-p)) = \ell(p) = 1$, since any nonconstant $f \in \mathfrak{L}(p)$ would give an isomorphism between X and $\mathbb{P}^1$ making $g = 0$. Also recall that $\deg K = 2g-2$. Putting these into the Riemann-Roch formula gives $\ell(K-p) = g-1 = \ell(K) - 1$. Now using the first part we know that ϕ_K defines a map into projective space.
(see Exercise 5.4 on page 124, and also remark 3.3 on page 133.)

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5) Let C be a canonical curve in $\mathbb{P}^n$, and H a hyperplane section, i.e. H is a divisor obtained by intersecting C with a hyperplane. Recall that the usual map $r : \mathbb{C}[x_0, \ldots, x_n] \to \bigoplus_{k=0}^{\infty} \mathfrak{L}(kH)$ is surjective. Assume that C has genus $g = 5$. Show that there are three irreducible quadratic hypersurfaces Q_1, Q_2, Q_3 in $\mathbb{P}^4$ such that $C \subset Q_1 \cap Q_2 \cap Q_3$.

Since C is a canonical divisor, any hyperplane section is a canonical divisor. Then $\ell(H) = g = 5$, and from the Riemann-Roch formula $\ell(kH) = (2k-1)(g-1) = 4(2k-1)$ for $k > 1$. If S^k denotes the space of all homogeneous polynomials in the variables $x_0, \ldots, x_4$, then $\dim S^k = \begin{pmatrix} 4+k \\ k \end{pmatrix}$. Since the map r is surjective, the space of homogeneous polynomials of degree k vanishing identically on C has dimension equal to $\dim S^k - \ell(kH)$. This dimension is zero for $k = 1$ and three for $k = 2$. The quadrics forming the basis define quadric hypersurfaces Q_1, Q_2, Q_3 and C is in their intersection. These quadrics are irreducible since otherwise C would lie in a hyperplane which is not the case.
(see Problem 6-a on page 211, and also check the arguments on pages 148-150 for the dimension count. See the arguments on page 146 for the canonical divisor of a canonical curve. The definition of canonical curve is on page 135.)
