Date: March 19, 2007, Monday NAME:..
Time: 10:00-12:00
Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Midterm Exam I

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

Q-1) Show that every fundamental open set $D_f \subset \mathbb{A}^n$ is isomorphic to an affine variety by constructing the isomorphism.

Solution: Let $f \in k[x_1, \ldots, x_n]$. We define the fundamental open set D_f as

$$D_f = \{p \in \mathbb{A}^n \mid f(p) \neq 0\}.$$

Consider the map

$$\begin{array}{rcl} \phi : D_f : & \longrightarrow & \mathbb{A}^{n+1} \\ (x_1, \ldots, x_n) & \mapsto & (x_1, \ldots, x_n, \dfrac{1}{f(x_1, \ldots, x_n)}). \end{array}$$

Observe that ϕ is a regular invertible map whose inverse is also regular, so ϕ defines an isomorphism onto its image. The image on the other hand is the zero set of the polynomial

$$1 - x_{n+1} f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_{n+1}].$$

Thus D_f is isomorphic to the hypersurface $Z(1 - x_{n+1} f(x_1, \ldots, x_n)) \subset \mathbb{A}^{n+1}$.

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Q-2) Construct a nontrivial k-algebra morphism ϕ from $k[x,y,z]/(x^2+y^2+z^2)$ to $k[u,v]$ and construct a regular map $f : \mathbb{A}^2 \longrightarrow Z(x^2+y^2+z^2) \subset \mathbb{A}^3$ such that $\phi = f^*$.

Solution: One possible solution may be as follows:

Consider the map $F : k[x,y,z] \longrightarrow k[u,v]$ defined as a k-algebra morphism by

$$F(x) = u^2 - v^2, \quad F(y) = iu^2 + iv^2, \quad F(z) = 2uv.$$

Since $\ker F \supset (x^2+y^2+z^2)$, F extends to a map $\phi : k[x,y,z]/(x^2+y^2+z^2) \longrightarrow k[u,v]$. Now define f as $f(u,v) = (u^2-v^2, iu^2+iv^2, 2uv)$. Clearly $f(\mathbb{A}^2) \subset Z(x^2+y^2+z^2)$ and $f^* = \phi$.

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Q-3) For a morphism $f : X \longrightarrow Y$ of affine varieties, show that f^* is injective if and only if $f(X)$ is dense in Y.

Solution: Here f^* is understood to be the induced map on the affine coordinate rings. Assume $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$. Take any polynomial $g \in k[x_1, \ldots, x_m]$. Then $f^*(g) = g \circ f$ and $\ker f^* = I(f(X))$ by definitions. Now f^* is injective if and only if $\ker f^* = I(Y)$ by definition. On the other hand $I(Y) = I(f(X))$ if and only if $Y = Z(I(Y)) = Z(I(f(X)) = \overline{f(X)}$, and this last condition is precisely the definition of $f(X)$ being dense in Y.

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Q-4) Let $C \subset \mathbb{A}^2$ be the curve given as $Z(x^2 - y^2 - x^4)$. Find the singular points of C and resolve the singularities by blowing up.

Solution: Let $f(x, y) = x^2 - y^2 - x^4$. Then $(f_x, f_y) = (2x(1 - 2x^2), -2y) = (0, 0)$ if and only if $(x, y) = (0, 0)$ or $(x, y) = (\pm 1/\sqrt{2}, 0)$, but these latter points do not lie on the curve so the only singularity is at the origin.

Blowing up using $x = X$, $y = XY$, we get $x^2 - y^2 - x^4 = X^2(1 - Y^2 - X^2) = 0$. Here $X = 0$ gives the exceptional divisor. The blow up curve $1 - Y^2 - X^2 = 0$ is clearly smooth and intersects the exceptional divisor at the points $(0, 1)$ and $(0, -1)$ corresponding to the slopes of the branches of the original curve at the origin, which can be found by factoring $x^2 - y^2 = 0$.

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Q-5) Let X and Y be projective varieties and $k[X]$ and $k[Y]$ their homogeneous coordinate rings. Prove or disprove: X and Y are isomorphic as projective varieties if and only if $k[X]$ and $k[Y]$ are isomorphic as k-algebras.

Solution: This is one of the drastic differences between affine and projective geometries. The statement is clearly true for affine varieties but is false for projective varieties. In the affine case the ring of global regular functions on X is given by the affine coordinate ring and we use this to construct maps but in the projective case the global regular functions are only the constants so we cannot construct maps using them. The projective coordinate ring then does not relate to functions on X as intrinsically as it did in the affine case. Here is a simple example. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ be given by $[x : y] \mapsto [x^2 : y^2 : xy] = [u : v : w]$. Check that ϕ is an isomorphism of $\mathbb{P}^1$ onto its image whose coordinate ring is $k[u, v, w]/(uv - w^2)$. Check that this ring is not isomorphic to $k[x, y]$; assume $F : k[u, v, w]/(uv - w^2) \to k[x, y]$ is an isomorphism. Then $F(u), F(v), F(w)$ must have no common factors since u, v, w have no common factors, but $F(u)F(v) = F(w)^2$ which implies that either $F(u)$ or $F(v)$ has a common factor with $F(w)$.