Date: March 27, 2008, Thursday NAME:..
Time: 13:40-15:30
Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Midterm Exam I – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

Q-1) Show that, if $X \subsetneq Y$ are two affine varieties, then $\dim X < \dim Y$.

Solution: If $\dim X = n$, then there exists a maximal chain of closed subsets of X in the form

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X,$$

which gives

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X \subsetneq Y,$$

which in turn means $\dim X < \dim Y$.

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Q-2) Show that two affine varieties X and Y are isomorphic if and only if their affine coordinate rings, $k[X]$ and $k[Y]$, are isomorphic.

Solution: This is a classical result. See for example:
Miles Reid, *Undergraduate Algebraic Geometry*, page 70,
or
Hartshorne, *Algebraic Geometry*, page 20.

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Q-3) Show that the only global regular functions on a projective variety are the constants. Show also that a projective variety and an affine variety are isomorphic to each other if and only if each is a singleton.

Solution: I have done the first part in class in detail. The second part follows from the first part when you compose your isomorphism, or its inverse, with the coordinate functions of the affine variety.

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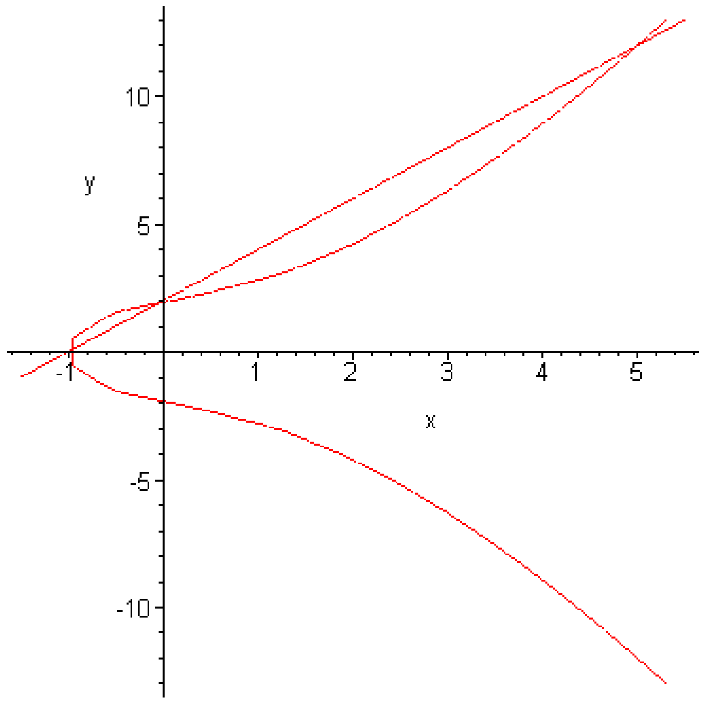
Q-4) Sketch the curve $y^2 = x^3+3x+4$ in $\mathbb{R}^2$. Using the group law on a cubic, find the sum of the points $(-1, 0)$ and $(0, 2)$ on this curve. (Actually everything holds in $\mathbb{P}^2$ over $\mathbb{C}$, but we are here calculating over the real image.)

Solution: The rule for such curves is that if P, Q, R are points on the curve, the $P + Q + R = 0$ if and only if P, Q, R are collinear, and if $R = (a, b)$, then $-R = (a, -b)$, see Miles Reid page 40. The line through the points $P = (-1, 0)$ and $Q = (0, 2)$ intersects the curve at $Q = (5, 12)$, and $P + Q = -R = (5, -12)$.



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Q-5) Sketch the curve $y^2 = x^3 + x^2$ over $\mathbb{R}^2$. Blow it up at its singularity and find the equations of its blow up at each of the coordinate charts in the blow up. Find the points where the resolved curve intersects the exceptional divisor. Do these intersection points have any relevance to the singularity of the curve?

Solution: This curve has a node type singularity at the origin where the two arms of the curve have $+1$ and -1 slope. After the blow up, in one of the charts the curve is of the form $Y^2 = X + 1$ which intersects the exceptional divisor $X = 0$ at $Y = \pm 1$ corresponding to the slopes of the arms of the curve at its singularity. The intersection points on the image of the exceptional divisor on the other chart should be reciprocals of these intersection points, so should still be ± 1.