Date: April 14, 2008, Monday NAME:..
Time: 13:40-15:30
Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

Q-1) The folium of Descartes, $x^3 + y^3 + 9xy = 0$, is a singular curve in $\mathbb{R}^2$. Find the point at which this curve meets the line at infinity in $\mathbb{P}^2_{\mathbb{R}}$. Also find the slope with which it intersects the line at infinity.

Solution:

First homogenize with respect to z: $x^3 + y^3 + 9xyz = 0$. With these coordinates, the line at infinity corresponds to $z = 0$. Intersecting it with our curve gives $x^3 + y^3 = 0$, or $x = -y$ since we are working over $\mathbb{R}$. Thus the point at infinity on this curve in $\mathbb{P}^2_{\mathbb{R}}$ is $[-1, 1, 0]$.

To find the slope, de-homogenize with respect to y by setting $u = x/y$ and $v = z/y$, to find $u^3+1+9uv = 0$. Consider v as a function of u and implicitly differentiate this equation with respect to u to obtain $3u^2 + 9v + 9uv' = 0$. The point at infinity corresponding to $(u, v) = (-1, 0)$ gives the slope at the intersection point as $v' = 1/3$.

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Q-2) Consider the hyperbola $x^2 - y^2 = 1$ and the line $y = \alpha x$ with $0 \leq \alpha < \infty$ in $\mathbb{R}^2$. Find their points of intersection, depending on α, in $\mathbb{P}^2_{\mathbb{R}}$.

Solution:

If $0 \leq \alpha < 1$, then the line and the hyperbola intersect in the affine plane when $x = \pm\sqrt{1/(1-\alpha^2)}$. There is no intersection when $\alpha > 1$ since we are working over the reals. When $\alpha = 1$ the intersection is on the line at infinity. Homogenize the hyperbola with respect to z and de-homogenize with respect to y to obtain $u^2 - 1 = v^2$ where $u = x/y$ and $v = z/y$. The line at infinity corresponds to $v = 0$, giving the intersection points as $[\pm 1 : 1 : 0]$ in $\mathbb{P}^2_{\mathbb{R}}$. These points correspond to the asymptotic lines $y = x$ and $y = -x$ of the given hyperbola.

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Q-3) Let $G = \{5x + 9y \mid x, y \in \mathbb{N}\}$. Show that G is not an Arf semigroup. Construct the Arf closure $^*G$ of G.

Solution:

$G = \{0, 5, 9, 10, 14, 15, 18, 19, 20, 23, 24, 25, 27, 28, 29, 30, 32 + \mathbb{N}\}$.
$G_1 = \mathrm{span}_{\mathbb{N}}\{0, 4, 5, 9, 10, 13, 14, 15, 18, 19, 20, 22, 23, 24, 25, 27 + \mathbb{N}\}$
$G_1 = \{0, 4, 5, 8, 9, 10, 12 + \mathbb{N}\}$.
$G_2 = \mathrm{span}_{\mathbb{N}}\{0, 1, 4, ...\}$
$G_2 = \mathbb{N}$.
$^*G = \{0, 5 + {}^*G_1\} = \{0, 5 + \{0, 4 + {}^*G_2\}\} = \{0, 5 + \{0, 4 + \mathbb{N}\}\} = \{0, 5, 9 + \mathbb{N}\}$.

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Q-4) Let H be the ring in the formal power series ring $\mathbb{C}[[t]]$ generated by elements of the form

$$\sum_{m,n\in\mathbb{N}} c_{m,n}(t^4)^m(t^{10}+t^{15})^n, \text{ where } c_{m,n}\in\mathbb{C}.$$

Show that H is not an Arf ring. Du Val calculated the Arf characters of the branch corresponding to H as $4, 10, 17$. Using this find the multiplicity sequence of this branch. In particular find out how many times you should blow up the singularity before it is resolved.

Solution:

Letting $X = t^4$ and $Y = t^{10} + t^{15}$, we observe that $[I_4]$ contains the element $(Y/X)^2 - (X^2/X)^3 = 2t^{17} + t^{22}$, but the set I_4/t^4 does not contain any element of order 17. Hence H is not an Arf ring.

Applying the Du Val-Jacobi algorithm to the set of Arf characters we find that the multiplicity sequence is $4, 4, 2, 2, 2, 2, 1, ...$, so we need to blow up 6 times to resolve the singularity.

Here is the Du Val-Jacobi algorithm applied to Arf characters:

Smallest element of $\{4, 10, 17\}$ is 4 and goes 2 times into the second smallest element 10. This gives $m_1 = m_2 = 4$.

The next step starts with $\{4, 10-8, 17-8\} = \{4, 2, 9\}$. Repeating the above procedure, we find $m_3 = m_4 = 2$.

The next step starts with $\{4-4, 2, 9-4\}$ or after omitting zero $\{2, 5\}$. This gives $m_5 = m_6 = 2$.

The next step starts with $\{2, 5-4\} = \{2, 1\}$. Since this gives $m_7 = m_8 = 1$, the singularity is resolved, and we stop.

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Q-5) Let H be a subring of the formal power series ring $\mathbb{C}[[t]]$ satisfying the condition that if $W(H) = \{\mathrm{ord}\phi \mid \phi \in H\} = \{i_0 = 0, i_1, i_2, \dots\}$, with the understanding that $0 < i_n < i_{n+1}$ for every $n = 1, 2, \dots$, then for every $n = 1, 2, \dots$, there exists and element $S_{i_n} \in H$ with $\mathrm{ord}S_{i_n} = i_n$ such that every element of H is in the form $\sum_{\ell=0}^{\infty} c_\ell S_{i_\ell}$ with $c_\ell \in \mathbb{C}$.
Show that if $\alpha = 1 + c_1 S_{i_1} + c_2 S_{i_2} + \cdots$ is in H, then its inverse, $1/\alpha$ is also in H.

Solution:

We can set $\beta_1 = -c_1$ so that

$$\alpha\,(1 + \beta_1 S_{i_1}) \equiv 1 \mod t^{i_1+1}.$$

Assume we found $\beta_1, \dots, \beta_{n-1} \in \mathbb{C}$ such that

$$\alpha \left(\prod_{\ell}^{n-1} (1 + \beta_\ell S_{i_\ell}) \right) \equiv 1 \mod t^{i_{n-1}+1}.$$

In other words

$$\alpha \left(\prod_{\ell}^{n-1} (1 + \beta_\ell S_{i_\ell}) \right) = 1 + \gamma_n S_{i_n} + \gamma_{n+1} S_{i_{n+1}} + \cdots .$$

Then setting $\beta_n = -\gamma_n$ we can see that

$$\alpha \left(\prod_{\ell}^{n} (1 + \beta_\ell S_{i_\ell}) \right) \equiv 1 \mod t^{i_n+1}.$$

Hence

$$1/\alpha = \prod_{\ell=1}^{\infty} (1 + \beta_\ell S_{i_\ell}) \in H.$$