Date: May 24, 2010, Monday
Time: 12:15-14:15
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 431 Algebraic Geometry – Final Exam – Solutions

1	2	3	4	5	*Bonus*	TOTAL
20	20	20	20	20	*+20*	100

Please do not write anything inside the above boxes!

Check that there are 5+1 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

In this exam we are working on an algebraically closed field k.

Q-1) Define the blow-up of $\mathbb{A}^2$ at the origin and apply it to show that a blow up resolves the singularities of the plane curve $y^2 = x^3 + x^2$.

Solution:

The blow up of the affine plane at the origin is the space

$$\{((x,y),[u;v]) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xv = yu\ \}.$$

Blowing up the above curve and changing to local coordinates gives the following equations:
$y^2 - (yx)^3 - (yx)^2 = y^2(1 - yx^3 - x^2) = 0$. Here $y^2 = 0$ gives the exceptional divisor and $1 - yx^3 - x^2 = 0$ gives a smooth curve.
$(yx)^2 - x^3 - x^2 = x^2(y^2 - x - 1) = 0$. Here again $x^2 = 0$ gives the exceptional divisor and $y^2 - x - 1 = 0$ gives a smooth curve. Hence, one blow up resolves the singularity.

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Q-2) Let $X = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 \mid x_1^3 = x_0 x_2^2 \}$. Show that X is a surface birational to $\mathbb{P}^2$.

Solution:

Check that the rational maps $[x_0 : x_1 : x_2 : x_3] \mapsto [x_1 : x_2 : x_3]$ and $[u : v : w] \mapsto [u^3 : uv^2 : v^3 : v^2 w]$ are inverses of each other and gives the birational isomorphism between X and $\mathbb{P}^2$.

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Q-3) For a smooth algebraic curve C and a divisor $D \in Div(C)$, define the space $L(D)$ and show that it is a vector space over k of finite dimension.

Solution:

$$L(D) = \{f \in k(C) \mid (f) + D \geq 0 \} \cup \{0\}.$$

That $L(D)$ ia a vector space follows directly from the properties of the ord_P function which is a valuation. In particular the fact that $ord_P(f+g) \geq \min\{ord_P(f), ord_P(g)\}$ forces $f+g$ to be in $L(D)$ when both are.

Let $\dim_k L(D) = \ell(D)$.

It is easy to show that if D_1 is linearly equivalent to D_2, then the vector spaces $L(D_1)$ and $L(D_2)$ are isomorphic. Moreover, since $\deg((f)+D) = \deg D \geq 0$ for $f \in L(D)$, it follows that $\ell(D) = 0$ when $\deg(D) < 0$.

If there are no effective divisors equivalent to D, then $L(D) = \{0\}$.

Assume now without loss of generality that D is effective.

If $\deg D = 0$, then since D is effective, $D = 0$ and $L(D) = k$, and consequently $\ell(D) = 1$.

Let $d = \deg(D) \geq 1$. Choose points $p_1, \ldots, p_{d+1}$ on C not lying on the support of D. Consider the map

$$\begin{aligned} \phi : L(D) &\rightarrow k^{d+1} \\ f &\mapsto (f(p_1), \ldots, f(p_{d+1})) \end{aligned}$$

Observe that $\ker \phi = L(D - p_1 - \cdots - p_{d+1}) = \{0\}$ since the degree of the divisor involved is negative. But the kernel has at most codimension $d+1$ in $L(D)$, which holds when ϕ is surjective. This gives $\ell(D) \leq d+1$ and hence $L(D)$ is finite dimensional.

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Q-4) Let C be a smooth algebraic curve and $D \in Div(C)$ with $\deg(D) = 0$ and $D \neq 0$. Prove or disprove that there exists a non-zero rational function f on C with $(f) = D$ if and only if $\ell(D) > 0$.

Solution:

The statement is true and we will prove it.

First assume that $\ell(D) > 0$. Let $g \in L(D)$ be non-zero and such that $(g) + D \geq 0$. But since $\deg((g) + D) = \deg(D) = 0$, we must have $(g) + D = 0$, which gives $D = (f)$ where $f = 1/g$.

Next assume that there exists a non-zero rational function f on C with $(f) = D$. Then it is clear that $1/f$ is non-zero and is in $L(D)$. Hence $\ell(D) > 0$.

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Q-5) Define what it means for a curve to be *hyperelliptic*. Prove or disprove that every smooth algebraic curve of genus 2 is hyperelliptic.

Solution:

A smooth curve C is called *hyperelliptic* if $g(C) \geq 2$ and there is a surjective morphism from C onto $\mathbb{P}^1$ of degree 2.

The statement is true and we will prove it.

Let C be a genus 2 curve with canonical divisor K. Since $\ell(K) = 2$, we may take K to be effective. Since $\deg K = 2g-2 = 2$, we may take $K = p+q$ for some points p and q on C. Take a non-constant f in $L(D)$, which is possible since $\ell(D) = 2$. Then the map $x \mapsto [1 : f(x)]$ is the map onto $\mathbb{P}^1$ which makes C hyperelliptic.

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Q-B) Let C be a smooth curve lying in $\mathbb{P}^n$ and satisfying the property that every hyperplane in $\mathbb{P}^n$ intersects C in exactly n points, counting multiplicities. Prove or disprove that C is isomorphic to $\mathbb{P}^1$.

Solution:

The statement is true and we will prove it.

Let f be a linear polynomial in $k[x_0, \dots, x_n]$ and $D = C \cap Z(f)$ where $Z(f)$ is as usual the zero set of the polynomial f in $\mathbb{P}^n$. Note that $\deg(D) = n$.

The rational functions $x_0/f, \dots, x_n/f$ are linearly independent and are all in $L(D)$. Hence $\ell(D) \geq n + 1$. By Riemann-Roch theorem we have

$$\ell(D) = n + 1 + (\ell(K - D) - g),$$

so we must have $\ell(K - D) \geq g$. But $\ell(K - D) \leq \ell(K) = g$, so we must have $\ell(K - D) = g$.

If $g \geq 2$, then $\ell(K - D) \leq \ell(K - p) = g - 1$ for some p in the support of D, since K is base point free. But this contradicts $\ell(K - D) = g$.

If $g = 1$, then $\deg(K - D) = -n$, so $\ell(K - D) = 0$, which contradicts $\ell(K - D) = g$.

Hence $g = 0$ and C is rational.