Date: March 5, Friday
Time: 08:40-10:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 431 Algebraic Geometry – Midterm Exam I – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

In this exam we are working on an algebraically closed field k.

Q-1) Show that the union of two affine algebraic varieties is again an algebraic variety.

Solution:
A proof is given on page 18 of the textbook. Here is another proof.

Assume $p \in V(J_1) \subset V(J_1) \cup V(J_2)$. This means that for any $f \in J_1 \cap J_2 \subset J_1$, $f(p) = 0$, implying that $p \in V(J_1 \cap J_2)$.

Conversely assume that $p \in V(J_1 \cap J_2)$. If $p \in V(J_1)$, we are done. If not, then there exists $g \in J_1$ such that $g(p) \neq 0$. Take any $f \in J_2$. Then $fg \in J_1 \cap J_2$, so by assumption about p we must have $f(p)g(p) = 0$. But since g does not vanish at p, we have $f(p) = 0$, implying that $p \in V(J_2) \subset V(J_1) \cup V(J_2)$.

We showed that $V(J_1) \cup V(J_2) = V(J_1 \cap J_2)$, which means that the union of two varieties is agaian variety.

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Q-2) Assuming that for any $n \geq 1$, $V(J) \neq \emptyset$ for any proper ideal $J \subset k[x_1, \ldots, x_n]$, show that $I(V(J)) \subseteq \sqrt{J}$.

Solution:
This is the standard *Rabinovitsch* trick, see page 25 of the textbook.

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Q-3) Let $X = V(y - x^2)$, $Y = V(xy - 1) \subset \mathbb{A}^2$. Show that X and Y are not isomorphic to each other. Show however that the projective closures $\overline{X}$ and $\overline{Y}$ in $\mathbb{P}^2$ are isomorphic.

Solution:

Since $k[X] \simeq k[x]$ and $k[Y] \simeq k[x, 1/x]$, these two varieties are not isomorphic. The projective closures are given by $\overline{X} = V(yz - x^2)$ and $\overline{Y} = V(xy - z^2)$ in $\mathbb{P}^2$. Clearly the map $[x : y : z] \mapsto [z : y : x]$ is an isomorphism between the two.

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Q-4) Let $X = V(x^2y + xy^2 - x^4 - y^4) \subset \mathbb{A}^2$. Show that X is singular at the origin and has three branches entering the origin with their corresponding tangents having equations $x = 0$, $y = 0$ and $x + y = 0$. Blow up X at the origin. Does this resolve the singularity? How many more blow-up operations do you need to resolve the singularity?

Solution:
The tangents directions at the origin are given by $x^2y + xy^2 = 0$, which gives the above directions.

To blow this up at the origin, put $x = X$, $y = XY$ to obtain $(Y - X) + Y^2 - XY^4 = 0$ which is smooth. Note that this curve intersects the exceptional divisor $Z = 0$ at $Y = 0$, $Y = -1$ and $Y = \infty$ corresponding to the slopes of the tangents of the branches of the original curve at the origin.

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Q-5) Using the definition that a line tangent at infinity to a plane curve is an asymptotic line to that curve and using the projective closure of $\mathbb{A}^2$ in $\mathbb{P}^2$, find the equations of all asymptotic lines for the curve C given by $xy + x^3 + y^3 = 0$ if it has an asymptotic line. Is there a real asymptote, i.e. when $k = \mathbb{R}$?

Solution:

Projective closure $\overline{C}$ of C is given by $xyz+x^3+y^3 = 0$. Dehomogenize by $y = 1$ to get $xz+x^3+1 = 0$. The point at infinity corresponds to $z = 0$ which forces $x \in \{-1, \exp(\pi/3),\ \exp(-\pi/3)\}$. The gradient of $xz+x^3+1 = 0$ is $(z+3x^2, x)$, and at $(\omega, 0)$ it is $(3\omega^2, \omega)$, where $\omega \in \{-1, \exp(\pi/3),\ \exp(-\pi/3)\}$. The equation of the tangent line at infinity is then given by $(3\omega^2, \omega)\cdot(x-\omega, z) = 0$, or $3\omega^2 x+3+\omega z = 0$. Homogenizing with respect to y and dehomogenezing with respect to z gives the equations of the asymptotic lines as $3\omega^2 + 3y + \omega = 0$. Putting $\omega = -1$ gives the asymptote in $\mathbb{R}^2$.