Due Date: 27 May 2011, Friday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Final Take-Home Exam – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are 4 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
Also note that if you write down something which you don't believe yourself, the chances are that I will not believe it either.

NAME: STUDENT NO:

Q-1) Show that every smooth non-degenerate curve $C \subset \mathbb{P}^{g-1}$ of genus g and degree $2g-2$ is a canonical curve.

Solution:

Let D be a hyperplane section of C. Then $D > 0$ and $\deg D = 2g - 2$. Moreover $D = H \cap C$ where H in $\mathbb{P}^{g-1}$ is given by $\lambda_1 x_1 + \cdots + \lambda_g x_g = 0$.

Consider the functions $f_\alpha = \dfrac{x_\alpha}{\lambda_1 x_1 + \cdots + \lambda_g x_g} \in K(C)$, $\alpha = 1, \dots, g$. Since $(f_\alpha) = (x_\alpha) - (\lambda_1 x_1 + \cdots + \lambda_g x_g) = (x_\alpha) - D$, we have $f_\alpha \in \mathcal{L}(D)$, $\alpha = 1, \dots, g$.

The functions $f_1, \dots, f_g$ are linearly independent, otherwise there would be constants $c_1, \dots, c_g$, not all zero, such that $c_1 f_1 + \cdots + c_g f_g = 0$, implying that C lies in the hyperplane $c_1 x_1 + \cdots + c_g x_g = 0$ contradicting the assumption that C is non-degenerate. Therefore $\ell(D) \geq g$.

By Riemann-Roch theorem, $\ell(D) = \deg D - g + 1 + i(D) = g - 1 + \ell(K - D)$ where $K = (\omega)$ is a canonical divisor of C, where ω is a holomorphic 1-form on C. This implies that $\ell(K - D) \geq 1$. Let $f \in \mathcal{L}(K - D)$ be a non-zero element. Then $(f) + (\omega) - D \geq 0$. But since $\deg((f) + (\omega) - D) = 0$, we must have $(f) + (\omega) - D = 0$, so $(f\omega) = D$. Thus D is the divisor of a holomorphic 1-form $D = (f\omega) = (\omega_1)$, setting $\omega_1 = f\omega$.

For every $g \in \mathcal{L}(D)$ we have $(g) + D = (g\omega_1) \geq 0$, so $g \mapsto g\omega_1$ is an injective morphism from $\mathcal{L}(D)$ to $\Omega^1(C)$. Considering dimensions, we see that this injection is an isomorphism. So $f_1\omega_1, \dots, f_g\omega_1$ is a basis of $\Omega^1(C)$. Consider the corresponding canonical map

$$
\begin{array}{rcl}
\phi_K : C & \longrightarrow & \mathbb{P}^{g-1} \\
[x_1 : \cdots : x_g] & \mapsto & [f_1\omega_1 : \cdots : f_g\omega_1] \\
& = & [f_1 : \cdots : f_g] \\
& = & [\dfrac{x_1}{\lambda_1 x_1 + \cdots + \lambda_g x_g} : \cdots : \dfrac{x_g}{\lambda_1 x_1 + \cdots + \lambda_g x_g}] \\
& = & [x_1 : \cdots : x_g] = identity.
\end{array}
$$

Hence ϕ_K is an isomorphism and $\phi_K(C) = C$, thus showing that C is a canonical curve.

Remark: This is problem 7 on page 211. See also the hint and remark that follows.

NAME: STUDENT NO:

Q-2) Let C be a compact Riemann surface and $D \geq 0$ be an effective divisor with $\ell(D) > 0$ and $\ell(K - D) > 0$. Show that $\ell(D) \leq \frac{1}{2} \deg D + 1$.

Solution:

For any divisor D, let $|D|$ denote the set of all effective divisors linearly equivalent to D. Then $|D|$ is isomorphic to the projectivization of $\mathcal{L}(D)$, hence it is a projective space of dimension $\ell(D) - 1$.

Let D_1 and D_2 be any two effective divisors. Consider the map

$$\begin{array}{rcl} \alpha : |D_1| \times |D_2| & \longrightarrow & |D_1 + D_2| \\ (A, B) & \mapsto & A + B. \end{array}$$

Given any $D \in |D_1 + D_2|$, there are only finitely many different ways of writing D as $A + B$ where A and B are effective divisors. And only some, if any, of them satisfies $A \in |D_1|$ and $B \in |D_2|$. So α is a finite-to-one mapping and this gives $\dim |D_1| + \dim |D_2| \leq \dim |D_1 + D_2|$.

Letting $D_1 = D$ and $D_2 = K - D$, we see that $\ell(D) + \ell(K - D) \leq g + 1$. But Riemann-Roch gives $\ell(D) - \ell(K - D) = \deg D - g + 1$. Adding these two side by side we get $2\ell(D) \leq \deg D + 1$ as required.

Remark: This is known as Clifford's Theorem. it is Exercise 4.4 on page 188. See also the extensive hint given there.

NAME: STUDENT NO:

Q-3) Let C be a compact complex Riemann surface and D any divisor on C. Show, using the basic definitions, that $\mathcal{L}(D)$ is finite dimensional and in fact that $\ell(D) \leq \deg D + 1$ when $\deg D \geq 0$. (Clearly $\ell(D) = 0$ when $\deg D < 0$.)

Solution:

We will first show that if D is an effective divisor, then $\ell(D) \leq \deg D + 1$. We will prove this statement by induction on the degree of the effective divisor D.

If $\deg D = 0$, then $D = 0$ and in this case $\mathcal{L}(D) \cong \mathbb{C}$, so $\ell(D) = 1$ and the above statement holds trivially.

Now assume that the statement holds for all effective divisors D of degree $\leq n$ for some non-negative integer n. Let p be any point on our curve C. Consider the divisor $D + p$. Let z be a local coordinate on C around p such that p corresponds to $z = 0$. For any $f \in \mathcal{L}(D+p)$, let $m = \nu_p(f)$ be the order of vanishing of f at p. And let $c_f^{(m)}$ be the residue of f/z^{m+1} at p. This means that the Laurent expansion of f at p, in terms of z is given by $f(z) = c_f^{(m)} z^m + c_f^{(m+1)} z^{m+1} + \cdots$. This defines a linear map

$$\begin{aligned} \phi : \mathcal{L}(D+p) &\rightarrow \mathbb{C} \\ f &\mapsto c_f^{(m)}, \ \text{ where } \ m = \nu_p(f). \end{aligned}$$

The kernel of this map is $\mathcal{L}(D)$, and $\ell(D+p) \leq \ell(D) + 1$ where equality holds if and only if ϕ is onto. Now using our induction assumption on $\ell(D)$, we conclude that $\ell(D+p) \leq n+2$ as required. This completes the induction and hence the proof that for every effective divisor D we must have $\ell(D) \leq \deg D + 1$.

Now let D be an arbitrary divisor on C with $\deg D \geq 0$. If $\ell(D) = 0$, then $\ell(D) \leq \deg D + 1$ holds trivially.

If $\ell(D) > 0$, then there is an $f \in \mathcal{L}(D)$ such that $(f) + D \geq 0$ by definition. Call $D' = (f) + D$. Clearly $D' \sim D$ and $\ell(D') = \ell(D)$ and $\deg D' = \deg D$. Since $D' \geq 0$, we can use the above statement that $\ell(D') \leq \deg D' + 1$, which proves the same statement for D.

Remark: See also the excellent argument given on page 102 which shows that $\ell(D) \leq \deg D + 1$ for an effective divisor D.

Here is another popular argument. Say $\deg D = n > 0$ and take $p \in C$ not in the support of D. Consider the map

$$\begin{aligned} \phi : \mathcal{L}(D) &\rightarrow \mathbb{C}^{n+1} \\ f &\mapsto (f(p), f'(p), \ldots, f^{(n)}(p)). \end{aligned}$$

This is a linear map whose kernel is precisely $\mathcal{L}(D - (n+1)p)$. Since $\deg D - (n+1)p < 0$, we have $\ell(D - (n+1)p) = 0$. It now follows that ϕ is injective and hence $\ell(D) \leq n + 1$.

NAME: STUDENT NO:

Q-4) Calculate the intersection number of the curves $x^3 - x^2 + y^2 = 0$ and $y^3 - x^2 = 0$ at the origin. Rotate the first curve by $\pi/4$ degrees counterclockwise and calculate its intersection number with $y^3 - x^2 = 0$ at the origin. Explain what happened geometrically.

Solution:

Parametrize $y^3 - x^2 = 0$ by $x = t^3$, $y = t^2$. Substitute this into $x^3 - x^2 + y^2 = 0$ to obtain $t^4 - t^6 + t^9 = 0$ which vanishes to order 4 at the origin. hence the intersection number at the origin is 4.

Rotating the first curve by $\pi/4$ we get something like $(y - x)^3 - 4\sqrt{2}\,xy = 0$. Doing the same parametrization here gives vanishing to order 5. Hence the intersection number in this case is 5.

In the first case each branch of $x^3 - x^2 + y^2 = 0$ at the origin is like a straight line passing through the origin of $y^3 - x^2 = 0$ and intersecting it in another point. Since the total intersection number must be three, the origin supplies two fold intersection. Two branches supplying intersection number two each makes four.

In the second case, one branch, $y = 0$, intersects the cusp $y^3 - x^2 = 0$ at the origin and at infinity, $[x : y : z] = [1 : 0 : 0]$, thus giving intersection number two at the origin. But the other branch, $x = 0$, intersects the cusp only at the origin so supplies a multiplicity of three. Adding up two and three gives five.

Remark: This is problem 5 on page 208.