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Math 114 Algebraic Geometry – Homework 1 – Solutions

Textbook: Phillip A. Griffiths, Introduction to Algebraic Curves, AMS Publications, 1989.

Q-1) Exercise 3.1 (page 14) Prove that the definition of $\nu_p(F)$ is well defined.

Solution:

Suppose C is a compact Riemann surface with $f \in K(C)$, $f \neq 0$ and $p \in C$. Select a local coordinate z around p such that $z(p) = 0$. Then around p

$$f = z^\nu h(z)$$

where $h(z)$ is a holomorphic function around p and $h(0) \neq 0$ with $\nu \in \mathbb{Z}$. This ν is defined to be $\nu_p(f)$.

Assume w is another local coordinate around p with $w(p) = 0$.

For a point q in a neighborhood of p where both z and w are defined, we have

$$f(q) = f \circ z^{-1}(z(q)) = (z(q))^\nu (h \circ z^{-1}(z(q))).$$

Set $\alpha = (z \circ w^{-1})$. This is a biholomorphic function and

$$z(q) = \alpha(w(q)) = a_1 w(q) + a_2 w(q)^2 + \cdots$$

is the Taylor extension with $a_1 \neq 0$. We have no constant term since $0 = z(p) = \alpha(w(p)) = \alpha(0)$. Putting this into the expression for f we have

$$\begin{aligned}
f(q) &= (f \circ z^{-1})(z(q)) \\
&= (z(q))^\nu (h \circ z^{-1}(z(q))) \\
&= (a_1 w(q) + a_2 w(q)^2 + \cdots)^\nu (h \circ z^{-1}(z \circ w^{-1} \circ w(q))) \\
&= (w(q))^\nu (a_1 + a_2 w(q) + \cdots)^\nu (h \circ w^{-1}(w(q))) \\
&= (w(q))^\nu (H \circ w^{-1}(w(q)))
\end{aligned}$$

where $H(q) = g(q)h(q)$ with $g(q) := (a_1 + a_2 w(q) + \cdots)^\nu$ is holomorphic around p with $H(p) \neq 0$. This shows that $\nu_p(f) = \nu$ and is independent of the holomorphic coordinate chosen.

Q-2) Exercise 3.2 (page 14) Suppose that C is either the Riemann sphere S or the complex torus $\mathbb{C}/\Lambda$. Take $f \in K(C)$. Verify that

$$\sum_{p \in C} \nu_p(f) = 0.$$

Solution:

First observe that zeros and poles of a rational function are isolated and when C is compact their total number is finite. So the sum in question is a finite sum.

Next assume that $C = S$. Take any point p_0 on S which is neither a zero nor a pole for F. Let γ be a small circle around p_0, oriented positively and containing no pole or zero of f on its interior. Since in general

$$\int_\gamma \frac{f'}{f} = \#(\text{zeros of } f \text{ inside } \gamma) - \#(\text{poles of } f \text{ inside } \gamma)$$

this integral is zero. On the other hand $S \setminus \{p_0\}$ is isomorphic to $\mathbb{C}$, and considering the same integral with reverse orientation we get

$$\begin{aligned} \int_{-\gamma} \frac{f'}{f} &= -\int_\gamma \frac{f'}{f} \\ &= \#(\text{poles of } f \text{ inside } -\gamma) - \#(\text{zeros of } f \text{ inside } -\gamma) \\ &= -\sum_{p \in C} \nu_p(f) \end{aligned}$$

since we count the zeros and poles with multiplicity. But now this integral and hence the sum is zero.

Next let $C = \mathbb{C}/\Lambda$. This time we cannot play the above game since the complement of a point is not isomorphic to $\mathbb{C}$ but there is another game to be played.

Let γ be the boundary of a fundamental region for C in $\mathbb{C}$ oriented positively. On one hand we have

$$\int_\gamma \frac{f'}{f} = \#(\text{zeros of } f \text{ inside } \gamma) - \#(\text{poles of } f \text{ inside } \gamma).$$

On the other hand this integral is zero since the opposite sides of γ are identified but traversed in opposite directions thus causing a cancelation.

Q-3) Prove Remark 3.9 (page 15): Clearly any meromorphic function f on a Riemann surface C is a holomorphic mapping into the Riemann sphere S.

Solution:

Using the notation on pages 6 and 7, define a map

$$\alpha : C \longrightarrow S$$

as follows. For any $p \in C$,

$$\alpha(p) = \begin{cases} \Phi_1^{-1}(f(p)) & \text{if } f(p) \neq \infty \\ \Phi_0^{-1}(1/f(p)) & \text{if } f(p) \neq 0. \end{cases}.$$

To check that this is well defined, it suffices to recall that

$$\Phi_1^{-1}(f(p)) = \Phi_0^{-1} \circ (\Phi_0 \circ \Phi_1^{-1})(f(p)) = \Phi_0^{-1}(1/f(p)).$$

It is now immediate to see that α is holomorphic.

Q-4) Show that the Riemann sphere S and the complex projective line $\mathbb{P}^1$ are isomorphic.

Solution:

We use the notation on page 7. Define a map $\alpha : S \longrightarrow \mathbb{P}^1$ as follows.

$$\alpha(X,Y,Z) = \begin{cases} [\frac{X+iY}{1-Z} : 1] & \text{if } Z \neq 1, \\ [1 : \frac{X-iY}{1+Z}] & \text{if } Z \neq -1. \end{cases}$$

Also define a map $\beta : \mathbb{P}^1 \longrightarrow S$ as follows.

$$\beta([s:t]) = \begin{cases} \Phi_1^{-1}(s/t) & \text{if } t \neq 0, \\ \Phi_0^{-1}(t/s) & \text{if } s \neq 0. \end{cases}$$

It is straightforward to check that these maps are well defined, holomorphic and are inverses of each other.
