Due Date: March 9, 2011 Wednesday NAME:..

MATH 431 Algebraic Geometry – Midterm Exam 1 – Solutions

Q-1) Show that $\mathbb{C}^n$ is not isomorphic to any $\mathbb{P}^m$. Also show that $\mathbb{P}^n$ is not isomorphic to any $\mathbb{P}^m$ with $n \neq m$. Here all spaces are considered as complex manifolds and isomorphisms are that of complex manifolds..

Solution:

If $\phi : \mathbb{P}^m \to \mathbb{C}^n$ is an isomorphism and $f : \mathbb{C}^n \to \mathbb{C}$ is a non-constant holomorphic function, then $f \circ \phi$ is a global non-constant holomorphic function on the compact space $\mathbb{P}^m$. Such a function cannot exist, so ϕ cannot exist.

Let $\phi : \mathbb{P}^n \to \mathbb{P}^m$ be an isomorphism. Let $p \in \mathbb{P}^n$ and $q = \phi(p) \in \mathbb{P}^m$. It is straightforward to show that the tangent spaces $T_p\mathbb{P}^n$ and $T_q\mathbb{P}^m$ are isomorphic as complex vector spaces. But two vector spaces are isomorphic if and only if their dimensions match.

Q-2) Consider the plane curve

$$y^2 + c_1xy + c_2y = x^3 + c_3x^2 + c_4x + c_5$$

where $c_i \in \mathbb{C}$ are some fixed constants. Show that by some change of coordinates, we can express this curve as

$$Y^2 = X(X-1)(X-\lambda)$$

where $\lambda \neq 0, 1$. Show that this curve is smooth, even at infinity. Show that this curve is not isomorphic to $\mathbb{P}^1$.

Solution:

The first part of the problem is elementary algebra. You may need to assume that the curve is smooth. To get a taste of the algebraic manipulations required, you may look at Hartshorne's *Algebraic Geometry*, page 319.

For the second part, call the curve C and assume that $\phi : \mathbb{P}^1 \to C$ is an isomorphism. Then $\phi = (f, g)$ where f and g are rational functions on $\mathbb{P}^1$, i.e. $f, g \in \mathbb{C}(t)$, satisfying $g^2 = f(f-1)(f-\lambda)$ with $\lambda \neq 0, 1$. That this leads to a contradiction is a slightly tricky calculation for which you may want to look at Hulek's *Elementary Algebraic Geometry*, pages 6-7.

The smoothness will follow from the usual calculations when you take $\lambda \neq 0, 1$.
