Due Date: May 26, 2014 Monday, 17:30 NAME:..

Instructor: Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Final Exam – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Let

$$H = k + kt^{4\nu}(1+t) + kt^{6\nu}(1+t) + kt^{7\nu}(1+t) + k[[t]]t^{8\nu},$$
$$H' = k + kt^{4\nu}(1+t+t^2) + kt^{6\nu}(1+t+t^2) + kt^{7\nu}(1+t+t^2) + k[[t]]t^{8\nu}$$

where $\nu > 2$. Show that these two rings are both Arf rings, have the same characters but are not isomorphic.

Solution:

These rings have the same characters which are

$$4\nu, 6\nu, 7\nu, 8\nu + 1.$$

The rings H_1, H_1' are both identical to

$$k + kt^{2\nu} + kt^{3\nu} + k[[t]]t^{4\nu}.$$

On the other hand there exists no substitution of the form

$$(\alpha) \qquad t \to t(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots)$$

which transforms H to H'. In fact such a transformation which maps H to H' should map H_1 to H_1', i.e. onto itself. Now assuming that 2ν is not divisible by the characteristic of k, the substitutions of the form (α), which transform the ring

$$H_1 = k + kt^{2\nu} + kt^{3\nu} + k[[t]]t^{4\nu}$$

onto itself, are of the form

$$t \to t(\alpha_0 + \alpha_\nu t^\nu + \alpha_{2\nu} t^{2\nu} + \alpha_{2\nu+1} t^{2\nu+1} + \cdots)$$

none of which transforms the element

$$t^{4\nu} + t^{4\nu+1}$$

of H to an element of the same order in H' which is of the form

$$\xi_0(t^{4\nu} + t^{4\nu+1} + t^{4\nu+2}) + \xi_2(t^{6\nu} + t^{6\nu+1} + t^{6\nu+2}) + \cdots .$$

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Q-2) For an effective divisor $D \geq 0$, on a curve X, define

$$|D| = \{D' \in \mathrm{Div}(X) \mid D' \geq 0 \text{ and } D' \sim D\},$$

where $D' \sim D$ means that there exists a rational function f on X such that $D' = D + (f)$.

(i) Show that $|D|$ is isomorphic to $\mathbb{P}^\ell$, where $\ell = \ell(D) - 1$.

(ii) For two effective divisors $D \geq 0$ and $E \geq 0$, show that

$$\dim |D| + \dim |E| \leq \dim |D + E|.$$

(iii) Prove Clifford's theorem that if D is an effective divisor such that $K - D$ is also effective, where K is the canonical divisor of X, then

$$\ell(D) \leq \frac{1}{2} \deg D + 1.$$

Solution:

There is a map

$$\begin{aligned} \phi : \mathcal{L}(D) &\to |D|, \\ f &\mapsto (f) + D. \end{aligned}$$

We see that $\phi(f) = \phi(g)$ if and only if $f = cg$ for some constant $c \in k$. Therefore $|D|$ is the projectivization of $\mathcal{L}(D)$. Therefore $\dim |D| = \ell(D) - 1$.

There is a map

$$\begin{aligned} \psi : &|D| \times |E| \to |D + E|, \\ (D', E') &\mapsto D' + E'. \end{aligned}$$

This map is finite to one since there are only finitely many ways of writing an effective divisor, such as $H \in |D+E|$ as a sum of two effective divisors, such as $D' + E'$. Hence the dimension of $\psi(|D| \times |E|)$ inside $|D + E|$ is precisely $\dim |D| + \dim |E|$, and this proves the second claim. We will use this claim in the form

$$\ell(D) + \ell(E) \leq \ell(D + E) + 1.$$

Now for Clifford's theorem: Take $E = K - D$, and use the above form of the inequality together with the Riemann Roch theorem, keeping in mind that $\ell(K) = g$, the genus of X.

$$\begin{aligned} \ell(D) + \ell(K - D) &\leq \ell(K) + 1 \\ \ell(D) - \ell(K - D) &= \deg D + 1 - g. \end{aligned}$$

Adding these lines side by side we get Clifford's theorem.

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Q-3) Assume that the projection $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is a closed map, i.e. it maps closed sets to closed sets (in the Zariski topology). Let $f : X \to Y$ be a morphism of projective varieties, $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$. Let $\Delta(Y) = \{(y, y) \in Y \times Y\}$ be the diagonal. Show that $\Delta(Y)$ is closed in $\mathbb{P}^m \times \mathbb{P}^m$. Let $\Gamma_f = \{(x, f(x)) \in X \times Y\}$ be the graph. Show that Γ_f is closed in $\mathbb{P}^n \times \mathbb{P}^m$. Now show that $f(X)$ is closed in Y. In particular show that if X and Y are curves (smooth and irreducible), then $f(X)$ is either Y or a point.

Solution:

Let $[y_0 : \cdots : y_m : y'_0 : \cdots : y'_m]$ be the homogeneous coordinates in $\mathbb{P}^m \times \mathbb{P}^m$. Let $I_Y \in k[y_0, \ldots, y_m]$ be the homogenous ideal of Y. Let $I'_Y \in k[y'_0, \ldots, y'_m]$ be the ideal consisting of all homogeneous polynomials $f(y'_0, \ldots, y'_m)$ where $f(y_0, \ldots, y_m) \in I_Y$. Finally let $J \in k[y_0, \ldots, y_m, y'_0, \ldots, y'_m]$ be the bihomogeneous ideal generated by elements of I_Y, elements of I'_Y and the elements $y_0 - y'_0, \ldots, y_m - y'_m$. Then clearly $\Delta(Y)$ is the zero set of J in $\mathbb{P}^m \times \mathbb{P}^m$.

Let the morphism $\phi : X \times Y \to Y \times Y$ be defined by $\phi(x, y) = (f(x), y)$. Then $\Gamma_f = \phi^{-1}(\Delta(Y))$, and being the inverse image of a closed set is itself closed.

Now we have $\Gamma_f \subset X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$ and $\pi(\Gamma_f)$ is closed in $\mathbb{P}^m$ by the assumption and hence is closed in $Y \in \mathbb{P}^m$. But $\pi(\Gamma_f)$ is nothing but $f(X)$.

The last claim follows from noting that the only closed irreducible subsets of a curve are the singletons and the curve itself.

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Q-4) Show that every projective algebraic set can be written as the zero set of finitely many homogeneous polynomials all of the same degree.

Solution:

Let $X = Z(J)$ and let J be generated by the homogeneous polynomials $f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$ of degrees $d_1, \ldots, d_m$ respectively. Let d be an integer greater than each d_i. Consider the homogeneous ideal I generated by the following set of homogeneous polynomials of degree d each:

$$x_0^{d-d_1} f_1, \ldots, x_n^{d-d_1} f_1,$$
$$\ldots$$
$$x_0^{d-d_i} f_i, \ldots, x_n^{d-d_i} f_i,$$
$$\ldots$$
$$x_0^{d-d_m} f_m, \ldots, x_n^{d-d_m} f_m.$$

Clearly $X = Z(I)$.

Another solution is as follows: Let $d = \mathrm{lcm}(d_1, \ldots, d_m)$, and set $F_i = f_i^{d/d_i}$, for $i = 1, \ldots, m$. Then $\deg F_i = d$ for all i, and clearly $Z(I) = Z(F_1, \ldots, F_m)$.