Due Date: March 10, 2014 Monday NAME:..

Instructor: Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Midterm Exam 1 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties.

(a) Show that $X \times Y \subset \mathbb{A}^{n+m}$ with its induced topology is irreducible.

(b) Show that the coordinate ring $k[X \times Y]$ of $X \times Y$ is isomorphic to $k[X] \otimes_k k[Y]$.

(c) Show that $X \times Y$ is a product in the category of varieties.

(d) Show that $\dim X \times Y = \dim X + \dim Y$.

Answer:

(a)
Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$, $i = 1, 2$. We first show that $X = X_1 \cup X_2$ and X_1, X_2 are closed.

Assume that there is an $x_0 \in X$ such that x_0 is neither in X_1 nor in X_2. Then define the sets $Y_i = \{y \in Y \mid x_0 \times y \in Z_i\}$, $i = 1, 2$. Now $Y = Y_1 \cup Y_2$, and we show that Y_1, Y_2 are closed subsets of Y. Fix $i = 1, 2$. Let us use the notation $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ for coordinates in $\mathbb{A}^{n+m}$. Let J_i be the ideal in $k[y]$ generated by the polynomials of the form $f_i(x_0, y)$ for all $f_i(x, y)$ belonging to the ideal of Z_i in $k[x, y]$. Then the zero set of J_i is precisely Y_i, so Y_i is closed. But Y is irreducible, so $Y = Y_1$ or Y_2, which implies that x_0 is in X_1 or in X_2, which is in contradiction to the way x_0 was chosen. So no such x_0 can be chosen and $X = X_1 \cup X_2$.

Next we show that X_1 is closed. Let J_1 be the ideal in $k[x]$ generated by polynomials $f(x, y_0)$ where $f(x, y)$ is a polynomial vanishing on Z_1 and y_o is a point on Y. Then X_1 is the zero set of J_1 and is therefore closed. Similarly X_2 is closed. But X being irreducible either $X = X_1$ or $X = X_2$. But this implies that either $X \times Y = Z_1$ or $X \times Y = Z_2$, showing that $X \times Y$ is irreducible.

(b)
Define a map $\phi : A(X) \otimes_k A(Y) \to A(X \times Y)$ as $\phi(\sum f_i(x) \otimes g_i(y)) = \sum f_i(x) g_i(y)$. This is a ring homomorphism. It is onto since $\phi(x_i \otimes y_j) = x_i y_j$ and they generate the ring $A(X \times Y)$. Now let r be the smallest integer such that there exist $F = \sum_{i=1}^{r} f_i(x) \otimes g_i(y)$ with $\phi(F) = 0$. From the minimality of r, we see that the g_i are not in the ideal of Y, so there is a point $y_0 \in Y$ such that not all $g_i(y_0)$ are zero. Assume without loss of generality that $g_r(y_0) \neq 0$. Then $\sum_{i=1}^{r} g_i(y_0) f_i(x) = 0$ on X, and we have $f_r(x) = \sum_{i=1}^{r-1} [g_i(y_0)/g_r(y_0)] f_i(x)$. Then we get

$$F = \sum_{i=1}^{r-1} f_i \otimes \{g_i(y) + [g_i(y_0)/g_r(y_0)] g_r(y)\},$$

violating the minimality of r. This contradiction shows that ϕ is injective and hence is an isomorphism.

(c)
Let $\pi_X : X \times Y \to X$ be the projection on the first component. Let ϕ be a regular function on X. Then $(\pi_X^*(\phi))(x, y) = (\phi \circ \pi_X)(x, y) = \phi(x)$ is a regular function on $X \times Y$, so π_X is a morphism of varieties. Similarly $\pi_Y : X \times Y \to Y$ is a morphism. Let Z be a variety with morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$. Define $\phi Z \to X \times Y$ as $\phi(z) = (p_X(z), p_Y(z) \in X \times Y$. We then have $\pi_X \circ \phi)(z) = \pi_X(p_X(z), p_Y(z)) = p_X(z)$ and similarly $\pi_Y \circ \phi = p_Y$, making the given diagram commutative.

This also follows from the universal property of tensor products if we consider the corresponding maps on the coordinate rings.

(d)
This follows from the fact that the dimension of a variety is the Krull dimension of its coordinate ring as follows.

$$\begin{aligned}\dim X \times Y = \dim A(X \times Y) &= \dim A(X) \otimes_k A(Y) \\ &= \dim A(X) + \dim A(Y) = \dim X + \dim Y.\end{aligned}$$

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Q-2) Let $X \subset \mathbb{A}^n$ be an affine variety of dimension r. Let $H \subset \mathbb{A}^n$ be a hypersurface such that $X \not\subset H$. Show that every irreducible component of $X \cap H$ has dimension $r - 1$. Give an example where $X \cap H$ has more than one irreducible component.

Solution:

Let Z be an irreducible component of $X \cap H$. By the Affine Dimension Theorem of Hartshorne page 48 (Proposition 7.1), $\dim Z \geq r + (n-1) - n = r - 1$. If $\dim Z = r$, then Z being an irreducible closed subset of the variety X of the same dimension is equal to X. But this means that $X \cap H = X$ and hence $X \subset H$ which contradicts the choice of H. So $\dim Z = r - 1$.

For the second part consider the curve $X \subset \mathbb{A}^2$ given by $y = x^2$ and the hyperplane, i.e. line, H given by $y = 1$. Then $X \cap H = \{(1,1)\} \cup \{(-1,1)\}$, has two irreducible components.

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Q-3) Let $J \subset k[x_1, \ldots, x_n]$ be an ideal generated by r elements. Show that every irreducible component of $Z(J)$ has dimension greater than or equal to $n - r$. Give an example where $Z(J)$ has more than one irreducible component.

Solution:

We show this by induction on r. Let J be generated by the non-constant polynomials $f_1, \ldots, f_r$. When $r = 1$, the result follows from the fact that the dimension of a hypersurface is $n - 1$.

Assume that each irreducible component of $Z(f_1, \ldots, f_{r-1})$ has dimension $n - r + 1$.

Any irreducible component Y of $Z(f_1, \ldots, f_r)$ is of the form $X \cap Z(f_r)$ where X is an irreducible component of $Z(f_1, \ldots, f_{r-1})$. Then by the Affine Dimension Theorem of Hartshorne page 48 (Proposition 7.1), we have $\dim Y \geq \dim X + \dim Z(f_r) - n \geq (n - r + 1) + (n - 1) - n = n - r$, as required, where $\dim X \geq n - r + 1$ was the induction hypothesis.

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Q-4) Let $\phi : \mathbb{A}^1 \to \mathbb{A}^2$ be the map defined by $\phi(t) = (t^2, t^3)$. Show that ϕ is a homeomorphism of $\mathbb{A}^1$ onto the curve $X = Z(y^2 - x^3)$, but is not an isomorphism in the category of algebraic sets.

Solution:

Let (x_0, y_0) be a point on the curve C given by $y^2 = x^3$. Since k is algebraically closed, there is $t_0 \in k$ such that $x_0 = t_0^2$. Then $y_0 = \pm t_0^3$. Hence $\phi(t_0) = (x_0, y_0)$ if $y_0 = t_0^3$, and $\phi(-t_0) = (x_0, y_0)$ if $y_0 = -t_0^3$. This shows that ϕ is bijective. The Zariski topology on $\mathbb{A}^1$ and on C declare that irreducible closed sets are points. Since ϕ and ϕ^{-1} clearly send points to points, both are continuous, making ϕ bicontinuous.

However for ϕ to be an isomorphism, ϕ^* on the coordinate rings must be a ring isomorphism. The coordinate ring $A(C)$ of C is isomorphic to $k[x, y]/(y^2 - x^3)$ and is generated by the equivalence classes of x and y. But $\phi^*(x) = t^2$ and $\phi^*(y) = t^3$. The coordinate ring of $\mathbb{A}^1$ is $k[t]$ and t is not in the range of ϕ^*, so ϕ^* is not surjective, and hence is not an isomorphism. This shows that the bicontinuous map ϕ is not an isomorphism in the category of affine varieties.