Due Date: April 21, 2014 Monday NAME:..

Instructor: Ali Sinan Sertöz STUDENT NO:..

Math 431 Algebraic Geometry – Midterm Exam 2 – Solutions

1	2	3	4	TOTAL
50	50	-	-	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Let H be a subring of $k[[t]]$ which contains all formal sums of its elements. Let $W(H) = \{i_0, i_1, i_2, \dots\}$ be the semigroup of orders of elements in H, where we have $0 = i_0 < i_1 < i_2 < \cdots$. Show that for any choice of elements $S_{i_0}, S_{i_1}, S_{i_2}, \dots$ in H with $\operatorname{ord} S_{i_\ell} = i_\ell$, we have

$$H = \{\sum_{\ell=0}^{\infty} \alpha_\ell S_{i_\ell} \mid \alpha_\ell \in k\}.$$

Answer:

Let $S \in H$ be an arbitrary element of order $i_r \in W(H)$. Let $\alpha_i = 0$ for $i = 0, \dots, r-1$, and set $\alpha_r = \operatorname{lc}(S)/\operatorname{lc}(S_{i_r}$, where lc denotes the leading coefficient, i.e. if $S = \alpha_r t^{i_r} +$ higher degree terms in t, where $\alpha_r \neq 0$, then $\operatorname{lc}(S) = \alpha_r$. Then $S' = S - \sum_{\ell=0}^{r} \alpha_\ell S_{i_\ell}$ has order strictly larger than $\operatorname{ord} S$. Repeating this argument with S' we obtain the result.

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Q-2) For any fixed positive integer r, choose elements $T_1, \ldots, T_r \in k[[t]]$ such that $\operatorname{ord} T_r > 0$ and

$$T_i \in kT_{i+1} + kT_{i+1}T_{i+2} + \cdots + kT_{i+1} \cdots T_{r-1} + k[[t]]T_{i+1} \cdots T_r,$$

for $i = 1, \ldots, r-1$.
Show that the ring

$$k + kT_1 + kT_1T_2 + \cdots + kT_1 \cdots T_{r-1} + k[[t]]T_1 \cdots T_r$$

is an Arf ring and moreover every Arf ring H is of this form if $\gcd W(H) = 1$.

Solution:

First we prove that if H is an Arf ring and $T \in H$ is an element of positive order, say $\operatorname{ord} T = d$, then the ring $k + HT$ is also an Arf ring. In fact let I_m be the ideal of all elements in $k + HT$ of orders greater or equal to m, where $m \geq d$ is an integer. Any element of I_m is of the form $f_n T$ where $f_n \in H$ is an element of order n and $n \geq m - d$. Let $f_r \in H$ be an element of order r where $r = m - d$. Then any element of $I_m/(f_r T)$ is of the form $(f_n T)/(f_r T) = f_n/f_r$. But since H is an Arf ring, the set of such elements forms a ring. Hence $I_m/(f_n T)$ is a ring, and $k + HT$ is thus an Arf ring.

Applying this result repeatedly, we see that any ring of the form mentioned in the question is an Arf ring since the first ring $k[[t]]$ is trivially an Arf ring.

To show that any Arf ring H, where $\gcd W(H) = 1$, is of this form, simply observe that for any h, $[I_h]$ is an Arf ring and $H = k + [I_h]T$ where $T \in H$ is an element of order h. Now repeating this process for the Arf ring $[I_h]$ and continuing we arrive eventually at a ring of the form mentioned in the question.