Midterm-2 for Math 431 **Due Date: 6 May 2022 Friday**

Solution Key:

1: *Show, in as much detail as possible, why* $\mathbb{A}^2 - \{(0,0)\}$ *is not affine. Check Internet to see why this is not as trivial as it looks!*

Let $X = \mathbb{A}^2 - \{(0,0)\}$. Any global regular function on X is of the form $\dfrac{f(x,y)}{g(x,y)}$, where $f, g \in k[x,y]$. Here g is allowed to vanish only at the origin if at all. The zero locus of any polynomial is of codimension 1, so g cannot vanish only at the origin. This says that g is a nonzero constant. Then the ring of global regular functions on X is $k[x,y]$.

If X is affine, then its coordinate ring is its ring of global regular functions which is $k[x,y]$. Moreover any regular map from $\mathbb{A}^2$ to X is induced by a ring morphism between their coordinate rings, both of which is $k[x,y]$. The identity homomorphism between these rings then induces the identity map from $\mathbb{A}^2$ to X. But the origin is not mapped to X, giving us a contradiction. Therefore X is not affine.

Some elementary arguments can be found at:
StackExchange

Some more sophisticated arguments can be found at
mathoverflow

2: *Show that the quadric surface Q given by $xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$.*

We have shown that Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ which is in turn birational to $\mathbb{P}^2$. Since Q contains lines which do not intersect, it cannot be isomorphic to $\mathbb{P}^2$ where all lines intersect.

3: Show that every quadratic variety of dimension n is isomorphic to a quadratic hypersurface in $\mathbb{P}^{n+1}$.

Here by a quadratic variety we mean a variety of degree two. This is Exercise I.7.8 on page 55 of Hartshorne's *Algebraic Geometry*. You can prove this result easily by using the results of the previous two exercises.

Moreover you can directly use a proposition which says that every irreducible projective variety X lies in a linear space of dimension $< \dim X + \deg X$. (*This is proposition on page 252 of Encyclopedia of Mathematical Sciences, Algebraic Geometry I, edited by Shafarevich and published by Springer. You can legally download a pdf copy when you are logged on Bilkent. The proof is easy and explained there in sufficient detail.*) Using this proposition we see that every degree 2 variety of dimension n lies in $\mathbb{P}^m$ where $m < n+2$, which is what we want to establish.

It is also known that every smooth variety X of dimension n lies in some $\mathbb{P}^{2n+1}$. This can be easily seen to be true since the secant variety of X has dimension $\leq 2n+1$. Thus if $X \subset \mathbb{P}^N$ with $N > 2n+1$, there is a point $O \in \mathbb{P}^N$ which does not lie on any secant or tangent of X and hence a projection from O sends X into $\mathbb{P}^{N-1}$.

The above proposition involving the degree is sharper when the degree of the variety is small.