Homework-2 for Math 431 **Due Date: 8 April 2022 Friday**

Show that the degree of the Hilbert polynomial of a projective variety of dimension d is also d. There are several proofs of this on Internet and in several books written with different tastes. Study them and write your own solution.

Write your solutions on A4 papers and staple them together. Do not use other fancy covers!

Thank you.

Solution:
Let $X \subset \mathbb{P}^n$ be an algebraic variety with $X = Z(I)$ where $I \subset S = k[x_0, \ldots, x_n]$ is a homogeneous prime ideal.

If $I = (x_0, \ldots, x_n)$, then the claim is trivially true: dimension of the empty set and the degree of the zero polynomial are both considered to be -1.

Now assume $I \neq (x_0, \ldots, x_n)$. Without loss of generality assume that $x_0 \notin I$. Let H be the hyperplane defined as the vanishing of x_0.

Let
$$M = S/I,\ M' = (S/I)(-1),\ M'' = M/M'.$$
We have an exact sequence
$$0 \to M' \xrightarrow{x_0} M \to M'' \to 0.$$
Note that we also have the exact sequence
$$0 \to M' \xrightarrow{x_0} M \to S/(I + (x_0)) \to 0,$$
which then implies that
$$M'' \cong S/(I + (x_0)).$$
We then have
$$\text{Ann } M'' = \text{Ann } S/(I + (x_0)) = Z(I + (x_0)) = X \cap H.$$
By induction hypothesis we may assume that the degree of the Hilbert polynomial $P_{M''}$ of $X \cap H$ is $d - 1$, since $\dim Z \cap H = d - 1$.

From the firat exact sequence above we have the following relation among the corresponding Hilbert functions

$$\phi_{M''}(t) = \phi_M(t) - \phi_{M'}(t-1).$$

For $t \gg 0$ we have $\phi_{M''}(t) = P_{M''}(t)$. Hence we have for large enough t,

$$\Delta\phi_M(t-1) = P_{M''}(t).$$

As in the proof of Proposition I.7.3.b, we can construct a numerical polynomial P of degree d such that $f(t) = P(t)$ for large t.

This completes the induction and the proof of the claim.