Homework-5 for Math 431 **Due Date: 29 April 2022 Friday**

Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements. Explain why you need no nilpotent elements.

Write your solutions on A4 papers and staple them together. Do not use other fancy covers!

Thank you.

Solution:

Let B be generated as a k-algebra by the elements $f_1, \ldots, f_n \in B$. Consider the exact sequence

$$0 \to J \to k[x_1, \ldots, x_n] \xrightarrow{\phi} B \to 0,$$

where ϕ ia an algebra morphism defined as

$$\phi(x_i) = f_i, \; i = 1, \ldots, n,$$

and $J = \ker \phi$.

Suppose B has a nilpotent element $b \neq 0$ with $b^m = 0$ for some m. Since ϕ is onto, there exists an element $u \in k[x_1, \ldots, x_n]$ such that $b = \phi(u)$. Clearly $b \notin J$. But $0 = b^m = \phi(u)^m = \phi(u^m]$. This shows that while $u \notin J$, we have $u^m \in J$. Hence J is not a radical ideal and cannot be the ideal of an algebraic set.

Conversely assume that B has no nilpotent elements. Assume that $u^m \in J$ for some m. Then $0 = \phi(u^m) = \phi(u)^m$. But since B has no nilpotent elements, $\phi(u) = 0$, forcing $u \in J$. Hence J is a radical ideal and is the ideal of some variety. B being $k[x_1, \ldots, x_n]/J$ is then the coordinate ring of that variety.