Date: January 11, 2010, Monday NAME:..
Time: 15:30-17:30
Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Let $\{f_n\}$ be a sequence of entire functions which converges to the entire function f. Assume that each f_n is one-to-one. Show that either f is one-to-one or constant.

Solution:

Let $a \in \mathbb{C}$ be any point. We want to show that $z \neq a$ implies that $f(z) \neq f(a)$.

Define the entire functions $h_n(z) = f_n(z) - f_n(a)$ and $h(z) = f(z) - f(a)$. Clearly the sequence $\{h_n\}$ of entire functions converges to the entire function h on the set $G_a = \mathbb{C} \setminus \{a\}$. By the Corollary 2.6 (to Hurwitz's Theorem on page 152) either $h \equiv 0$ or else it never vanishes on G_a. If $h \equiv 0$ then f is constant and is equal to $f(a)$. If f never vanishes on G_a, then this means that $z \neq a$ implies that $f(z) \neq f(a)$. Since this is true for all $a \in \mathbb{C}$, f is one-to-one.

(This was Exercise 10 on page 154.)

Q-2) During the development of the product formula $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$, Conway uses the identity

$$\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n} = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right),$$

"... *because the terms of the infinite product can be rearranged.*" Explain why the terms can be rearranged.

Solution:

On compact subsets of $\mathbb{C}$, we have $|1-E_{p_n}(z/a_n)| \leq \left(\frac{r}{|a_n|}\right)^{p_n+1}$, see bottom of page 169 for details. Here $a_n = n$ and $p_n = 1$. Now use Corollary 5.6 on page 166 to conclude that the infinite product converges uniformly, i.e $\sum(E_1(z/n)-1)$ converges absolutely so its terms and hence the terms of the infinite product can be rearranged in any way to give the same value.

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Q-3) Prove Wallis's formula: $\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}$.

Solution:
In the product formula for $\sin \pi z$, see the previous question, put $z = 1/2$ and simplify.

(This was Exercise 4 on page 176.)

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Q-4) Using the definition of the Gamma function $\Gamma(z) = \dfrac{e^{-\gamma z}}{z} \displaystyle\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$ as a meromorphic function on $\mathbb{C}$, derive the Gauss's formula, for $z \neq 0, -1, -2, \ldots$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}.$$

Using Gauss's formula, derive the functional equation, for $z \neq 0, -1, -2, \ldots$

$$\Gamma(z+1) = z\Gamma(z).$$

Solution:
This is all on pages 177 and 178.

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Q-5) Prove that $\zeta^2(z) = \sum_{n=1}^{\infty} \frac{d(n)}{n^z}$ for $Re\, z > 1$, where $d(n)$ is the number of divisors of n and

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \text{ for } Re\, z > 1.$$

Solution:
Let $a_1, \ldots, a_k$ be all the distinct divisors of n, so that $d(n) = k$. For each a_j define $b_j = n/a_j$. Then

$$\begin{aligned} \zeta^2(z) &= \left(\cdots + + \frac{1}{a_j^z} + \cdots\right)\left(\cdots + \frac{1}{b_j^z} + \cdots\right) \\ &= \cdots + \left(\sum_{j=1}^{k} \frac{1}{a_j^z b_j^z}\right) + \cdots \\ &= \cdots + \frac{k}{n^z} + \cdots \end{aligned}$$

(This was Exercise 3 on page 194.)