Due Date: 22 November 2012, Thursday NAME:..

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Math 503 Complex Analysis – Exam 05

1	2	3	4	5	TOTAL
50	50	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Problem 40.2 on page 272 of course textbook:

> Suppose that $\phi(x, y)$ is a harmonic function in a simply connected domain S. [A domain is an open and connected set in $\mathbb{C}$.] Show that ϕ has a harmonic conjugate in S given up to an additive constant by
>
> $$\theta(x, y) = \int_{(x_0,y_0)}^{(x,y)} -\phi_y dx + \phi_x dy,$$
>
> where the point (x_0, y_0) is fixed in S and the integral is independent of the path. Furthermore, show that in this problem the [simply] connectedness property is crucial.

Solution: First we recall some facts from Calculus.

Let $f(x, y)$ be any C^1 function on S. By the Fundamental Theorem of Line Integrals, which is a demonstration of the Fundamental Theorem of Calculus, we have

$$f(x, y) - f(x_0, y_0) = \int_C \nabla f \cdot dr,$$

where C is any smooth curve in S joining (x_0, y_0) to (x, y).

Stokes' theorem states that if $F = (M, N, P)$ is a vector field in a simply connected domain, then the line integral

$$\int_b^a F \cdot dr$$

in independent of the path joining b to a in S if and only if $\operatorname{curl} F = (0, 0, 0)$. Here

$$\operatorname{curl} F = (P_y - N_z, M_z - P_x, N_x - M_y).$$

In particular when F is a plane field, then we have $M = M(x, y)$, $N = N(x, y)$ and $P = 0$, and the curl condition reduces to $N_x - M_y = 0$ only.

Finally, assume that the vector field $F = (M, N, P)$ is conservative in a simply connected domain S. Fix a point (x_0, y_0, z_0) in S. Then the function

$$f(x, y, z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} F \cdot dr$$

is well defined on S, and straightforward calculation shows that

$$\frac{\partial f}{\partial x} = M, \ \frac{\partial f}{\partial y} = N, \ \frac{\partial f}{\partial z} = P.$$

Now back to our problem. Consider the vector field $F = (-\phi_y, \phi_x, 0)$. Note that $\operatorname{curl} F = (0, 0, \phi_{xx} + \phi_{yy}) = (0, 0, 0)$ since ϕ is harmonic. Moreover, the line integral

$$\theta(x, y) = \int_{(x_0,y_0)}^{(x,y)} F \cdot dr$$

is well defined, i.e. independent of path, since the domain is simply connected. From Calculus we know now that

$$\frac{\partial \theta}{\partial x} = -\phi_y, \ \frac{\partial \theta}{\partial y} = \phi_x,$$

showing that θ is the harmonic conjugate of ϕ on S.

Here simply connectedness is crucial as shown by the example $\phi(x, y) = \log(x^2+y^2)$ on $S = \mathbb{C}-\{0\}$. Here ϕ is harmonic but is not the real part of any analytic function on S. Hence it has no harmonic conjugate in S.

NAME: STUDENT NO:

Q-2) Show that the only non-negative harmonic functions on $\mathbb{C}$ are the constants.

Solution:

Let $u(z) = u(x, y)$ be a non-negative harmonic function on $\mathbb{C}$. Let $z = re^{it}$ be an arbitrary point in $\mathbb{C}$. By the Harnack inequality, for every $R > r$, we have

$$\frac{R-r}{R+r}u(0) \leq u(z) \leq \frac{R+r}{R-r}.$$

Taking limits of all sides as R goes to infinity we see that

$$u(z) = u(0), \text{ for all } z \in \mathbb{C},$$

hence u is constant.

We can prove this without Harnack inequality as follows. Since the domain is simply connected, u has a harmonic conjugate v, and the entire function $f = u + iv$ is such that the real part is bounded from below by zero. Let $a < 0$ be any negative number. Since the real part of f is never negative, we have

$$|f(z) - a| > |a|/2 > 0, \text{ for all } z \in \mathbb{C}.$$

Therefore $1/(f(z) - a)$ is an entire function and satisfies

$$\left|\frac{1}{f(z) - a}\right| < \frac{2}{|a|}, \text{ for } z \in \mathbb{C},$$

and hence is constant by Liouville's theorem. This forces f and consequently u to be constant.