Due Date: 6 December 2012, Thursday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Exam 07

1	2	3	4	5	TOTAL
40	20	20	20	0	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Consider the infinite sum

$$1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots + (-1)^{n+1}\frac{1}{n^z} + \cdots$$

where z is to be considered as a complex variable. Find a region where this series converges. Discuss if it can be extended to all of complex numbers and if so what kind of a function we get.

Solution: Observe that when $z = x + iy$,

$$\left|\frac{\pm 1}{n^z}\right| = \frac{1}{n^x},$$

and hence the given series converges absolutely for $\operatorname{Re} z > 1$. There we observe that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = \left(1 - \frac{2}{2^n}\right) \zeta(z).$$

Since the right hand side is certainly a meromorphic function on $\mathbb{C}$, the left hand side also extends to a meromorphic function. Note however that

$$\zeta(z) = \frac{1}{z-1} + g(z)$$

around $z = 1$ where $g(z)$ is analytic in a neighborhood of $z = 1$. Thus we see by a simple calculation that

$$\lim_{z\to 1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = \lim_{z\to 1} \left(1 - \frac{2}{2^n}\right) \left(\frac{1}{z-1} + g(z)\right) = \lim_{z\to 1} \left(1 - \frac{2}{2^n}\right) \left(\frac{1}{z-1}\right) = \ln 2.$$

This shows that our infinite sum actually extends to an entire function.

NAME: STUDENT NO:

Q-2) Find the exact setting and the exact phrasing (in English) of the Riemann hypothesis as Riemann himself stated it.

Solution:

In his only number theory article, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse", Monatsberichte der Berliner Akademie, (1859), Riemann sets $s = \frac{1}{2} + it$, where t is a complex number, and defines the function

$$\xi(t) := \Gamma\left(\frac{s}{2} + 1\right)(s-1)\pi^{-s/2}\zeta(s).$$

Then he says

> ..., the function $\xi(t)$ can vanish only when the imaginary part of t lies between $\frac{1}{2}i$ and $-\frac{1}{2}i$. ... and it is very likely that all of the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation.

Note that the only zeros of ξ are contributed by ζ and when t is real s is on the critical line. This is the Riemann Hypothesis.

NAME: STUDENT NO:

Q-3) State, but do not prove, a few striking consequences of the Riemann Hypothesis.

Solution:

Probably the first result to quote is

$$|\pi(x) - Li(x)| < \frac{1}{8}\sqrt{x}\,\log x, \text{ for all } x \geq 2657,$$

from Wikipedia. Here $Li(x) = \int_0^x dt/\log(t)$, and $\pi(x)$ is the number of primes less than x.

For a comprehensive discussion see
`http://mathoverflow.net/questions/`
`17209/consequences-of-the-riemann-hypothesis`

NAME: STUDENT NO:

Q-4) "It is known that $\pi(x) < Li(x)$ for all $x \leq 10^{23}$, and no value of x is known for which $\pi(x) > Li(x)$." (From Wikipedia)
State the story related to this mystery, keeping in mind that the number of atoms in the observable universe is calculated to be between 10^{78} and 10^{82}.

Solution:

The difference $\pi(x) - Li(x)$ changes sign infinitely many times and the first time such a sign change is expected is the so called Skewes' number which is around $10^{10^{10^{34}}}$.