Due Date: 14 December 2012, Friday NAME:..

Please leave your homework in my mailbox until 17:30.

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Exam 08

1	2	3	4	5	TOTAL
100	0	0	0	0	100

Please do not write anything inside the above boxes!

Check that there is **1** question on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Define the complex Gamma function as

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt, \quad \text{for } \operatorname{Re} z > 0.$$

Show that for any non-negative integer k, the k-th derivative of the Gamma function is given by the formula

$$\Gamma^{(k)}(z) = \int_0^\infty e^{-t}t^{z-1}(\ln t)^k\,dt, \quad \text{for } \operatorname{Re} z > 0.$$

Solution: We will show that $\Gamma(z)$ is complex differentiable for $\operatorname{Re} z > 0$ using ϵ-δ arguments. For this we need a candidate for the derivative. We anticipate that the derivative operator will commute with the integral sign and hence that the k-th derivative of the Γ function will be

$$\int_0^\infty e^{-t}t^{z-1}(\ln t)^k\,dt \quad \text{for} \quad \operatorname{Re} z > 0,$$

where k is any non-negative integer. We will first show that these integrals exist and converge uniformly on compact subsets of $\operatorname{Re} z > 0$.

Throughout this section we fix the notation as $z = x + iy$ where x and y are real and $x > 0$.

Lemma 1 *Let k be any non-negative integer. For all $\operatorname{Re} z > 0$, the integrals*

$$\int_0^\infty e^{-t}t^{z-1}(\ln t)^k dt$$

exist for $\operatorname{Re} z > 0$, and converge uniformly on compact subsets.

Proof: The existence of this integral is equivalent to the possibility of making the values

$$\left|\int_0^b e^{-t}t^{z-1}(\ln t)^k dt\right| \quad \text{and} \quad \left|\int_R^\infty e^{-t}t^{z-1}(\ln t)^k dt\right|$$

arbitrarily small by convenient choices of b and R. We will show that this is possible.

For any z with $\operatorname{Re} z > 0$ choose $0 < \alpha < 1$ and $\beta > 1$ such that

$$0 < \alpha < x < \beta.$$

From the usual observations that

$$\lim_{t\to 0^+} t^{\alpha/2}(\ln t)^k = 0 \quad \text{and} \quad \lim_{t\to\infty} t^{-\beta}(\ln t)^k = 0$$

we can choose $\delta > 0$ and $R_1 > 0$ such that

$$\left|t^{\alpha/2}(\ln t)^k\right| < 1 \quad \text{for all} \quad 0 < t < \delta,$$

and

$$\left|t^{-\beta}(\ln t)^k\right| < 1 \quad \text{for all} \quad t \geq R_1.$$

Then for all $0 < b < \delta$,

$$\begin{aligned}
\left|\int_0^b e^{-t}t^{z-1}(\ln t)^k dt\right| &\leq \int_0^b e^{-t}t^{x-1}|(\ln t)^k| dt \\
&\leq \int_0^b e^{-t}t^{\alpha-1}|(\ln t)^k| dt \\
&= \int_0^b e^{-t}t^{(\alpha/2)-1}|t^{\alpha/2}(\ln t)^k| dt \\
&\leq \int_0^b e^{-t}t^{(\alpha/2)-1} dt,
\end{aligned}$$

and similarly for all $R \geq R_1$,

$$\begin{aligned}
\left|\int_R^\infty e^{-t}t^{z-1}(\ln t)^k dt\right| &\leq \int_R^\infty e^{-t}t^{x-1}|(\ln t)^k| dt \\
&\leq \int_R^\infty e^{-t}t^{\beta-1}|(\ln t)^k| dt \\
&= \int_R^\infty e^{-t}t^{(2\beta)-1}|t^{-\beta}(\ln t)^k| dt \\
&\leq \int_R^\infty e^{-t}t^{(2\beta)-1} dt.
\end{aligned}$$

Now for any $\epsilon > 0$ choose $0 < \Delta \leq \delta$ and $R_0 \geq R_1$ such that for all $0 < b < \Delta$ and for all $R \geq R_0$ we have

$$\int_0^b e^{-t}t^{(\alpha/2)-1} dt < \epsilon \quad \text{and} \quad \int_R^\infty e^{-t}t^{(2\beta)-1} dt < \epsilon,$$

which we know we can do since $\Gamma(\alpha/2)$ and $\Gamma(2\beta)$ exist. This proves the existence of the given integral.

If D is a compact subset of $\operatorname{Re} z > 0$, then the α and β of the above argument can be chosen such that $\alpha < x < \beta$ holds for all $z \in D$. Then the existence of Δ and R_0 depend on ϵ and D but not on individual z in D. This proves the uniform convergence on compact subsets. $\square$

Now that we know the existence of the candidate functions, we formulate what we want to prove. We want to show that for any non-negative integer k and for any z with $\operatorname{Re} z > 0$, we have

$$\Gamma^{(k)}(z) = \int_0^\infty e^{-t}t^{z-1}(\ln t)^k \, dt.$$

We already know that this holds for $k = 0$. Assume that it holds for some $k \geq 0$. We then want to prove it for $k+1$. In other words we want to show that for this fixed k and for any z with $\operatorname{Re} z > 0$, and for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all Δz with $0 < |\Delta z| < \delta$, we have

$$\left| \frac{\Gamma^{(k)}(z+\Delta z) - \Gamma^{(k)}(z)}{\Delta z} - \int_0^\infty e^{-t} t^{z-1} (\ln t)^{k+1} dt \right| < \epsilon,$$

or equivalently, using the induction hypothesis,

$$\int_0^\infty e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| dt < \epsilon. \tag{1}$$

We expect to achieve this by choosing $|\Delta z|$ small. Let us agree that Δz will be chosen such that $0 < |\Delta z| < x/2$. Throughout our analysis we may have to make it smaller if necessary but we will never make it larger than $x/2$.

We start with a technical lemma.

Lemma 2 *For any z with $\operatorname{Re} z > 0$, for any integer $k \geq 0$, for any $t > 0$, and for any Δz with $0 < |\Delta z| < x/2$, we have*

$$\left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| \leq \begin{cases} |\ln t|^{k+1}\ t^{-x/2} & \text{if } 0 < t \leq 1, \\ |\ln t|^{k+1}\ t^{x/2} & \text{if } t \geq 1. \end{cases}.$$

Proof: Observe that

$$t^{\Delta z} = e^{(\Delta z)(\ln t)} = \sum_{n=0}^\infty \frac{(\Delta z)^n (\ln t)^n}{n!}.$$

Putting this into the given expression and simplifying, we get

$$\begin{aligned}
\left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| &= \left| (\ln t)^{k+1} \sum_{n=1}^\infty \frac{(\Delta z)^n (\ln t)^n}{(n+1)!} \right| \\
&\leq \left| (\ln t)^{k+1} \right| \sum_{n=1}^\infty \frac{|\Delta z|^n |\ln t|^n}{(n+1)!} \\
&\leq \left| (\ln t)^{k+1} \right| \sum_{n=1}^\infty \frac{|\Delta z|^n |\ln t|^n}{n!} \\
&\leq \left| (\ln t)^{k+1} \right| \sum_{n=0}^\infty \frac{|\Delta z|^n |\ln t|^n}{n!} \\
&= \left| (\ln t)^{k+1} \right| e^{|\Delta z|\, |\ln t|} \\
&= \left| (\ln t)^{k+1} \right| e^{|\ln t^{|\Delta z|}|}.
\end{aligned}$$

When $0 < t \leq 1$, we have

$$e^{|\ln t^{|\Delta z|}|} = e^{\ln t^{-|\Delta z|}} = t^{-|\Delta z|} \leq t^{-x/2}.$$

On the other hand, when $t \geq 1$, we have

$$e^{|\ln t^{|\Delta z|}|} = e^{\ln t^{|\Delta z|}} = t^{|\Delta z|} \leq t^{x/2},$$

and this completes the proof. □

We now return to the integral in equation (1), and consider it in three pieces.

Lemma 3 *Fix an integer $k \geq 0$, any z with $\operatorname{Re} z = x > 0$, and any Δz with $0 < |\Delta z| < x/2$. For any $\epsilon > 0$, there exists a number a with $0 < a < 1$ such that for all c with $0 < c \leq a$, we have*

$$\int_0^c e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| \, dt < \epsilon/3.$$

Proof: Since $0 < t < 1$, using the fact that $e^{-t} < 1$ and the previous lemma, we can write

$$\begin{aligned} e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| &< t^{x-1} t^{-x/2} |\ln t|^{k+1} \\ &= t^{(x/4)-1} [t^{x/4} (\ln \frac{1}{t})^{k+1}]. \end{aligned}$$

Since

$$\lim_{t \to 0^+} [t^{x/4} (\ln \frac{1}{t})^{k+1}] = 0,$$

for this given $\epsilon > 0$, there exists a real number a with $0 < a < 1$ such that for all t with $0 < t < a$, we have

$$[t^{x/4} (\ln \frac{1}{t})^{k+1}] < \frac{\epsilon\, x}{12}.$$

Finally we have for all c with $0 < c \leq a$,

$$\begin{aligned} \int_0^c e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| \, dt &< \int_0^c t^{(x/4)-1} [t^{x/4} (\ln \frac{1}{t})^{k+1}] \, dt \\ &< \frac{\epsilon\, x}{12} \int_0^c t^{(x/4)-1} \, dt \\ &= \frac{\epsilon\, x}{12} \left(\left. \frac{t^{x/4}}{x/4} \right|_{t=0}^{t=c} \right) \\ &= \frac{\epsilon\, x}{12} \frac{c^{x/4}}{x/4} \\ &< \frac{\epsilon\, x}{12} \frac{1}{x/4} \\ &= \epsilon/3, \end{aligned}$$

as claimed. □

Next we consider the other end of the integral in equation (1).

Lemma 4 *Fix an integer $k \geq 0$, any z with $\operatorname{Re} z = x > 0$, and any Δz with $0 < |\Delta z| < x/2$. For any $\epsilon > 0$, there exists a number $b \geq 1$ such that for all r with $r \geq b$, we have*

$$\int_r^\infty e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| \, dt < \epsilon/3.$$

Proof: Using Lemma 2, we can write for all $t \geq 1$,

$$\begin{aligned} e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| &< e^{-t} t^{x-1} [t^{x/2} (\ln t)^{k+1}] \\ &= e^{-t/2} [e^{-t/2} t^{(3x/2)-1} (\ln t)^{k+1}]. \end{aligned}$$

On the other hand, since

$$\lim_{t \to \infty} [e^{-t/2} t^{(3x/2)-1} (\ln t)^{k+1}] = 0,$$

there exists for this given $\epsilon > 0$, a number $b \geq 1$ such that for all $t \geq b$, we have

$$[e^{-t/2} t^{(3x/2)-1} (\ln t)^{k+1}] < \epsilon/6.$$

Now for any such $r \geq b$, our integral can be estimated as follows.

$$\begin{aligned}
\int_r^\infty e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| dt &< (\epsilon/6) \int_r^\infty e^{-t/2}\, dt \\
&= (\epsilon/6)\ \left(2e^{-r/2}\right) \\
&< (\epsilon/6)\ (2) \\
&= \epsilon/3,
\end{aligned}$$

as claimed. $\square$

Now we begin to consider the middle part.

Lemma 5 *Let $[a, b] \subset (0, \infty)$. For any $\epsilon > 0$, there exists a $\delta > 0$ such that for every $t \in [a, b]$, and for every Δz, we have*

$$0 < |\Delta z| < \delta \implies \left| \frac{t^{\Delta z} - 1}{\Delta z} - \ln t \right| < \epsilon.$$

Proof: Let $m_0 = \max\{|\ln a|, |\ln b|\}$.

We start by the Taylor expansion of $\frac{t^{\Delta z}-1}{\Delta z} - \ln t$ as a function of Δz around 0.

$$\begin{aligned}
\left| \frac{t^{\Delta z} - 1}{\Delta z} - \ln t \right| &= \left| \sum_{n=1}^{\infty} \frac{(\ln t)^{n+1}}{(n+1)!} (\Delta z)^n \right| \\
&\leq \sum_{n=1}^{\infty} \frac{|\ln t|^{n+1}}{(n+1)!} |\Delta z|^n \\
&\leq \sum_{n=1}^{\infty} \frac{m_0^{n+1}}{(n+1)!} |\Delta z|^n
\end{aligned}$$

which is now a real analytic function of $|\Delta z|$ defined everywhere. In particular it vanishes at the origin and is continuous there. Hence a $\delta > 0$ exists such that for all $|\Delta z| < \delta$ we have

$$\sum_{n=1}^{\infty} \frac{m_0^{n+1}}{(n+1)!} |\Delta z|^n < \epsilon.$$

This then completes the proof. $\square$

Corollary 6 *Let $[a, b] \subset (0, \infty)$ and fix any non-negative integer k. For any $\epsilon > 0$, there exists a $\delta > 0$ such that for every $t \in [a, b]$, we have*

$$|\Delta z| < \delta \implies \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| < \epsilon.$$

Proof: Let $m_0 = \max\{|\ln a|, |\ln b|\}$. By the same idea as above we can choose $\delta > 0$ such that for all $0 < |\Delta z| < \delta$ we will have

$$\left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| \leq \left| \sum_{n=1}^{\infty} \frac{m_0^{k+n+1}}{(n+1)!} |\Delta z|^n \right| < \epsilon$$

which completes the proof. $\square$

Note that by putting $k = 0$ in the corollary we recover the lemma.

We have one last technical lemma for the middle part.

Lemma 7 *Fix an integer $k \geq 0$, any z with $\operatorname{Re} z = x > 0$, and any interval $[a, b] \subset (0, \infty)$ with $0 < a < 1 < b$. For any $\epsilon > 0$, there exists a $\delta > 0$ such that for all Δz with $0 < |\Delta z| < \delta$, we have*

$$\int_a^b e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| dt < \epsilon/3.$$

Proof: Choose $0 < \delta < (\operatorname{Re} z)/2$ such that for all $t \in [a, b]$ and for all $0 < |\Delta z| < \delta$ we have

$$\left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| < \frac{\epsilon}{3\Gamma(x)}$$

as in corollary (6) above. Here $0 < \delta < (\operatorname{Re} z)/2$ is needed to assure that $\operatorname{Re}(z + \Delta z) > 0$ so that $\Gamma(z + \Delta z)$ is defined. Then

$$\begin{aligned}
\int_a^b e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| dt &< \frac{\epsilon}{3\Gamma(x)} \int_a^b e^{-t} t^{x-1} dt \\
&< \frac{\epsilon}{3\Gamma(x)} \int_0^\infty e^{-t} t^{x-1} dt \\
&= \epsilon/3,
\end{aligned}$$

as claimed. $\square$

We can now proceed with the derivative of the Γ function.

Theorem 8 *Let k be any non-negative integer. For any $\operatorname{Re} z > 0$, we have*

$$\Gamma^{(k)}(z) = \int_0^\infty e^{-t} t^{z-1} (\ln t)^k dt.$$

Proof: We will prove this by induction on k. The case $k = 0$ is trivial at this point, since it describes the Gamma function itself. Assume that the claim holds for a certain $k \geq 0$. We will show that the $k + 1$ derivative of the Gamma function is of the required form. For this we want to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that for any Δz with $0 < |\Delta z| < \delta$, we will have inequality (1) which is

$$\int_0^\infty e^{-t} t^{x-1} \left| \frac{t^{\Delta z}(\ln t)^k - (\ln t)^k}{\Delta z} - (\ln t)^{k+1} \right| dt < \epsilon.$$

For this given $\epsilon > 0$, choose $0 < a < 1$ as in Lemma 3, choose $b > 1$ as in Lemma 4, and choose $\delta > 0$ as in Lemma 7. Now clearly for any Δz with $0 < |\Delta| < \delta$, the required inequality (1) holds, completing the induction. $\square$