Due Date: 21 December 2012, Friday NAME:..

Please leave your homework in my mailbox until 17:30.

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Exam 09

1	2	3	4	5	TOTAL
30	50	20	0	0	100

Please do not write anything inside the above boxes!

Check that there is **1** question on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Find an entire function $f(z)$ whose zero set is $\{n+in \in \mathbb{C} \mid n \in \mathbb{Z}\}$. (Give the most elementary example.)

Solution:

Let $a_0 = 0$, and $a_{2n-1} = n$, $a_{2n} = -n$ for $n = 1, 2, \ldots$. This defines the sequence

$$0, 1, -1, 2, -2, ..., n, -n, \ldots.$$

Observe that

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{1+p_n} = \sum_{n=1}^{\infty} \left(\frac{r}{\sqrt{2}n}\right)^{1+p_n}$$

converges for all $r > 0$ if $p_n = 1$ for all $n = 1, 2, \ldots$. Therefore

$$f(z) = z\prod_{n=1}^{\infty} E_1\left(\frac{z}{a_n}\right) = z\prod_{n=1}^{\infty}\left(1-\frac{z}{n+in}\right)e^{z/(n+in)} \cdot \left(1+\frac{z}{n+in}\right)e^{-z/(n+in)}$$

is such a function. Simplifying further we find

$$f(z) = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{(n+in)^2}\right) = z\prod_{n=1}^{\infty}\left(1+\frac{iz^2}{2n^2}\right).$$

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Q-2) Show that $\cos \pi z = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)$.

Solution: If you want to this from scratch, here is how it goes:
The zeros of $\cos \pi z$ are precisely the half integers $(2n+1)/2$, $n \in \mathbb{Z}$. Since

$$\sum_{n=-\infty}^{\infty}\left(\frac{r}{(2n+1)/2}\right)^{1+p_n}$$

converges for all $r > 0$ when $p_n = 1$ for all n, we can write

$$\cos \pi z = e^{g(z)} \prod_{n=-\infty}^{\infty} E_1\left(\frac{z}{(2n+1)/2}\right) = e^{g(z)} \prod_{n=-\infty}^{\infty}\left(1 - \frac{z}{(2n+1)/2}\right) e^{z/((2n+1)/2)},$$

for some entire function g. Noting that for $n = 1, 2, \ldots$,

$$\frac{2(-n)+1}{2} = -\frac{2(n-1)+1}{2},$$

we can rewrite $\cos \pi z$ as

$$\cos \pi z = e^{g(z)} \prod_{n=1}^{\infty}\left(1 - \frac{2z}{-2n+1}\right)\left(1 - \frac{2z}{2n-1}\right) = e^{g(z)} \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

Note that by setting $z = 0$, we immediately see that $g(0) = 0$.

Setting $f(z) = \cos \pi z$, we see that

$$\frac{f'(z)}{f(z)} = -\pi \tan \pi z = g'(z) - \sum_{n=1}^{\infty} \frac{2z}{\left(\frac{2k-1}{2}\right)^2 - z^2}.$$

By the Mittag-Leffler expansion theorem, we have

$$\tan z = 2z \sum_{k=1}^{\infty} \frac{1}{\left(\frac{2k-1}{2}\pi\right)^2 - z^2}.$$

This forces

$$g'(z) = 0,$$

hence $g(z)$ is constant. Since we found $g(0) = 0$, we see that $g(z) \equiv 0$. This finally gives

$$\cos \pi z = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right),$$

as required.

Here is a solution which uses the infinite product formula for the sine function:

$$\begin{aligned}
\cos \pi z &= \frac{\sin 2\pi z}{2 \sin \pi z} \\
&= \frac{2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{(2z)^2}{n^2}\right)}{2\pi z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right)} \\
&= \frac{\prod_{k=1}^{\infty}\left(1 - \frac{(2z)^2}{(2k)^2}\right) \prod_{k=1}^{\infty}\left(1 - \frac{(2z)^2}{(2k-1)^2}\right)}{\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right)} \\
&= \prod_{k=1}^{\infty}\left(1 - \frac{(2z)^2}{(2k-1)^2}\right)
\end{aligned}$$

as expected.

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Q-3) Let $a_n = 2n - 1$ for $n = 1, 2, \ldots$. Show that

$$\pi = 2 \prod_{n=1}^{\infty} \left(\frac{2n}{a_n} \frac{2n}{a_{n+1}} \right).$$

Solution:

Using the infinite product expansion of $\sin \pi z$, put $z = 1/2$ and observe that

$$1 = \sin \pi \frac{1}{2} = \pi \frac{1}{2} \prod_{n=1}^{\infty} \left(1 - \frac{(1/2)^2}{n^2} \right) = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(\frac{a_n}{2n} \frac{a_{n+1}}{2n} \right),$$

which is equivalent to what we want to prove. This result is known as Wallis' formula.