Due Date: 8 January 2015, Thursday NAME:..

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Math 503 Complex Analysis – Take-Home Final Exam – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Prove that $\dfrac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty} \dfrac{\Lambda(n)}{n^z}$ for $\operatorname{Re} z > 1$, where $\Lambda(n) = \log p$ if $n = p^m$ for some prime p and $m \geq 1$; and $\Lambda(n) = 0$ otherwise.

Solution:

We use Euler's Theorem 8.17 on page 193: If $\operatorname{Re} z > 1$, then

$$\zeta(z) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - p_n^{-z}} \right)$$

where $\{p_n\}$ is the sequence of prime numbers.

Taking logarithm of both sides we get

$$\log \zeta(z) = -\sum_{n=1}^{\infty} \log(1 - p_n^{-z}),$$

and taking derivatives we get

$$\begin{aligned}
\frac{\zeta'(z)}{\zeta(z)} &= -\sum_{n=1}^{\infty} (\log p_n) \frac{p_n^{-z}}{1 - p_n^{-z}} \\
&= -\sum_{n=1}^{\infty} (\log p_n)[p_n^{-z} + p_n^{-2z} + \cdots + p_n^{-mz} + \cdots] \\
&= -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\log p_n}{p_n^{mz}} \\
&= -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}
\end{aligned}$$

as required.

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Q-2) Show that $\Gamma'(1) = -\gamma$, where $\gamma = \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \log n\right]$ is the Euler constant.

Solution:

Use the definition $\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n}$, and write the logarithmic derivative of $\Gamma(z)$ to obtain

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(n+z)}$$

as obtained on page 179 using Exercise 5.10 of page 174 (which in turn is a direct application of Theorem 2.1 of page 151.) Now putting in $z = 1$ and recalling that $\Gamma(1) = 1$ gives the result. (You have to recall from calculus that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1,$$

using the technique of telescoping series.)

NAME: STUDENT NO:

Q-3) Show that $\pi = 2\prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}$.

Solution:

Using the Weierstrass product formula for sine function

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

put $z = 1/2$ and simplify.

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Q-4) Let f be an entire function and let $a, b \in \mathbb{C}$ such that $|a| < R$ and $|b| < R$. If $\gamma_R(t) = Re^{it}$ with $0 \le t \le 2\pi$, evaluate $\displaystyle\int_{\gamma_R} \frac{f(z)}{(z-a)(z-b)}\, dz$. Use this result to give another proof of Liouville's Theorem.

Solution:

Letting $F(z) = \dfrac{f(z)}{(z-a)(z-b)}$ we see that

$$\int_{\gamma_R} \frac{f(z)}{(z-a)(z-b)}\, dz = 2\pi i \left(\text{Res}(F,a) + \text{Res}(F,b)\right) = 2\pi i \frac{f(a)-f(b)}{a-b}.$$

Now take any $z_0 \in \mathbb{C}$. Set $b = z_0$ and $a = z_0 + \Delta$ where $|\Delta| < 1$. Choose $R > 0$ such that $|a| < R/2$ and $|b| < R/2$. Then for any $z \in \gamma_R$ we have

$$|z-a| > \frac{R}{2} \quad \text{and} \quad |z-b| > \frac{R}{2}.$$

Now assume that $|f(z)| \le M$ for all $z \in \mathbb{C}$. We then have

$$2\pi \left| \frac{f(z_0+\Delta) - f(z_0)}{\Delta} \right| = \left| \int_{\gamma_R} \frac{f(z)}{(z-z_0-\Delta)(z-z_0)}\, dz \right| \le \frac{2\pi R M}{R^2/4}.$$

We thus have for all $\Delta \in \mathbb{C}$ with $|\Delta| < 1$ and for all sufficiently large $R > 0$,

$$\left| \frac{f(z_0+\Delta) - f(z_0)}{\Delta} \right| \le \frac{4M}{R}.$$

Taking limits as $\Delta \to 0$ and $R \to \infty$ we find that

$$f'(z_0) = 0 \quad \text{for all} \quad z_0 \in \mathbb{C},$$

which implies that f is constant.

Observe that the above discussion shows that

$$\left| \frac{f(a) - f(b)}{a-b} \right| \le \frac{4M}{R}$$

for all large R. Now taking R to infinity shows that $f(a) = f(b)$, hence f is constant.

Similarly, if we take $a = b$, we find that

$$\int_{\gamma_R} \frac{f(z)}{(z-a)^2}\, dz = 2\pi i \ \text{Res}(F,a) = 2\pi i \ f'(a).$$

Now we can similarly show that $f'(a) = 0$ when f is bounded, which again shows that f is constant.