Due Date: 13 October 2014, Monday – Class time NAME:..

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Math 503 Complex Analysis – Homework 1 – Solutions

1	2	3	4	5	TOTAL
40	60	0	0	0	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Starting from the following basic facts that we proved in class

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \quad \text{for} \quad \theta \in \mathbb{R}, \\ \sin z &= \sin x \cosh y + i \cos x \sinh y, \quad \text{and} \\ \cos z &= \cos x \cosh y - i \sin x \sinh y, \quad \text{for} \quad x, y \in \mathbb{R}, \end{aligned}$$

show that

$$e^{iz} = \cos z + i \sin z, \quad \text{for} \quad z \in \mathbb{C}.$$

Also show that for any $z_1, z_2 \in \mathbb{C}$, we have the addition rules

$$\begin{aligned} \sin(z_1 + z_2) &= \sin z_1 \cos z_2 + \sin z_2 \cos z_1, \\ \cos(z_1 + z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2. \end{aligned}$$

Solution:

Recall that for real y, we have the usual definitions $\cosh y = \dfrac{e^y + e^{-y}}{2}$ and $\sinh y = \dfrac{e^y - e^{-y}}{2}$. It follows that $\cosh y - \sinh y = e^{-y}$.

Now back to the question.

$$\begin{aligned} e^{iz} &= e^{-y+ix} = e^{-y}e^{ix} \\ &= e^{-y}\cos x + ie^{-y}\sin x \\ &= (\cosh y - \sinh y)\cos x + i(\cosh y - \sinh y)\sin x \\ &= (\cos x \cosh y - i \sin x \sinh y) + (i \sin x \cosh y - \cos x \sinh y) \\ &= (\cos x \cosh y - i \sin x \sinh y) + i(\sin x \cosh y + i \cos x \sinh y) \\ &= \cos z + i \sin z. \end{aligned}$$

The summation formulas can be easily verified by expanding both sides into their respective real and imaginary parts. However here is a shorter proof which uses a major property of holomorphic functions: An entire function vanishing along the real line is identically zero.

For any fixed $s \in \mathbb{R}$, define the entire function

$$f(z) = \sin(z + s) - (\sin z \cos s + \sin s \cos z).$$

This function clearly vanishes for real z, and is therefore identically zero. Now for any fixed $z_0 \in \mathbb{C}$, define the entire function

$$g(z) = \sin(z_0 + z) - (\sin z_0 \cos z + \sin z \cos z_0).$$

We just proved that this function vanishes for all real z, hence it is identically zero, proving the summation formula for sine function. Taking the derivative of $g(z) \equiv 0$ with respect to z, we get the summation formula for cosine function.

NAME: STUDENT NO:

Q-2) Consider the function

$$z \mapsto w = z + \frac{1}{z}, \quad \text{for} \quad z \in \mathbb{C},\ z \neq 0.$$

Describe the mapping properties of this map. In other words define a Riemann surface S such that the map is one-to-one and onto S.

In particular find a contour C in the z-plane such that (a) it goes around the point $z = 1$ once and totally lies in the right hand plane $\operatorname{Re} z > 0$, and (b) its image can be easily described under the above map. Then describe its image. How many times does it go around the branch point $w = 2$?

Solution:

For this mapping it is convenient to use polar coordinates. Let $z =^r e^{i\theta}$.

$$w = z + \frac{1}{z} = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta = u + iv.$$

When $r > 0$ is fixed but is not 1, the image satisfies the relations

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1,$$

which is an ellipse with major axis $r + 1/r$, and minor axis $r - 1/r$. When $r = 1$, the ellipse collapses to the interval $[-2, 2]$. In fact in that case

$$u = 2\cos\theta,$$

and thus goes from $+2$ to -2 as θ goes from 0 to π. Then as θ goes from π to 2π, it traces the same interval back from -2 to $+2$, suggesting that we should have a cut along this interval. Notice that the circles $z = re^{i\theta}$ and $z = (1/r)e^{i\theta}$ map to the same ellipse, so they should be considered on different sheets of the Riemann surface for the image.

The derivative of $w = z + \frac{1}{z}$ is $w' = \frac{z^2 - 1}{z^2}$, and vanishes only at $z = \pm 1$. Thus at other points the map is conformal.

The rays through the origin, where θ is fixed are orthogonal to the circles around the origin and they should be sent to curves orthogonal to the ellipses which are images of these circles, except when the rays pass through ± 1.

When θ is fixed but is not a multiple of $\pi/2$, the image of the ray through the origin with angle θ satisfies the equation

$$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1,$$

which is a hyperbola orthogonal to each of the above ellipses. When $v = 0$, the hyperbola intersects the u-axis at the point $2\cos\theta \in (-2, 2)$. Thus it intersects the branch cut so must continue on the other sheet.

When $\theta = 0$, the image becomes

$$w = \left(r + \frac{1}{r}\right) \cos\theta.$$

Therefore the interval $(0, 1]$ is mapped onto the interval $[2, \infty)$, in reverse direction. The interval $[1, \infty)$ is mapped onto the same interval $[2, \infty)$ in increasing order. The angle between the intervals $(0,]1$ and $[1, \infty)$ at the point $z = 1$ is π. We write the Taylor series of $w - 2$ at $z = 1$ in terms of $z - 1$ as follows.

$$(w - 2) = (z - 1) - 1 + \frac{1}{1 + (z - 1)} = (z - 1)^2 - (z_1)^3 + \cdots,$$

which vanishes to order 2, so the angles at $z = 1$ are multiples by 2 at the point $w = 2$. This explains why the angle between the two images is 2π. The same phenomenon happens at $z = -1$. This time the intervals $[-1, 0)$ and $(-\infty, -1]$ map to the interval $(-\infty, -2]$ twice, making an angle of 2π at $w = -2$.

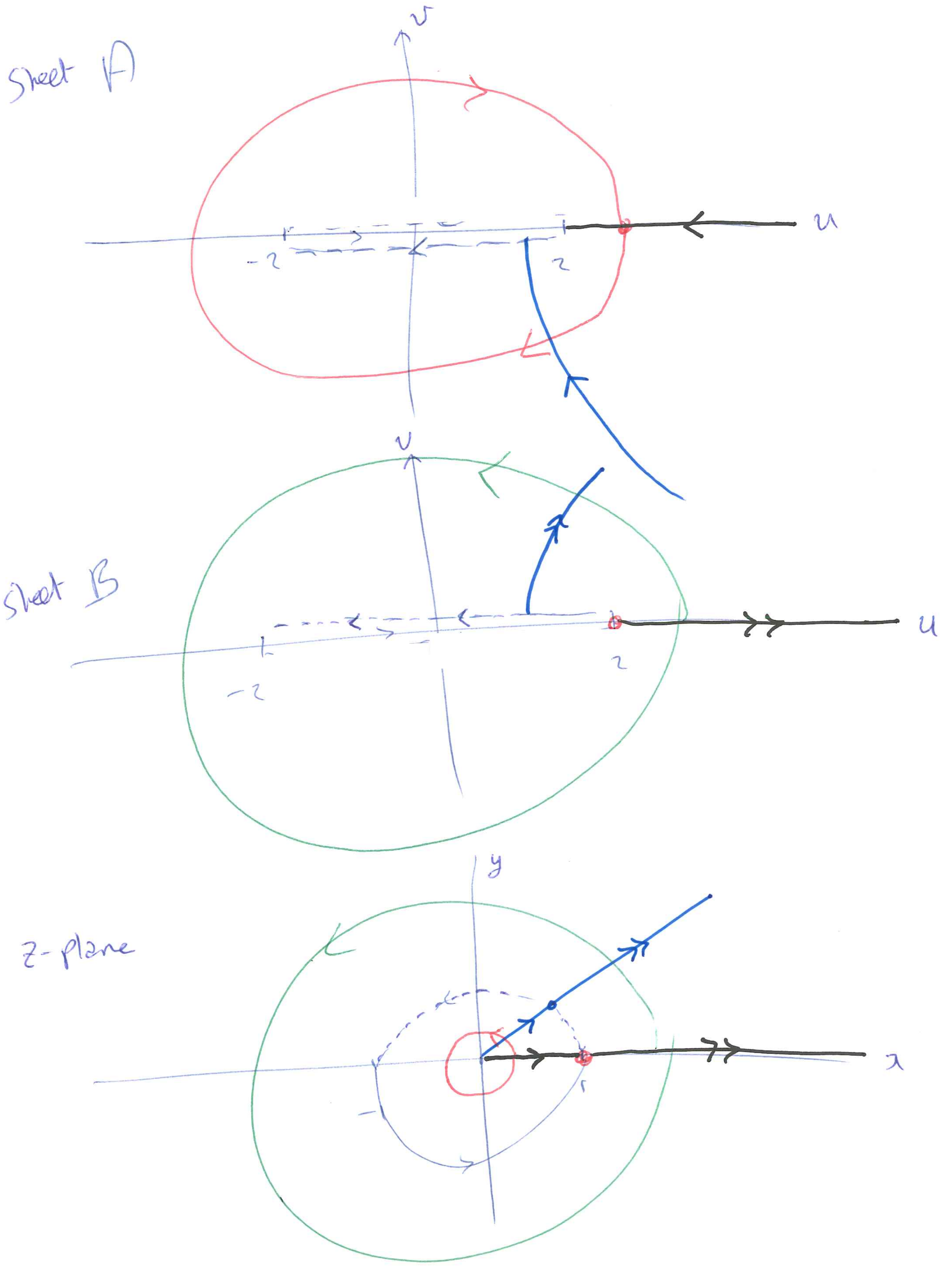
On the other hand when $\theta = \pi/2$, the image becomes

$$w = i\left(r - \frac{1}{r}\right) \sin\theta.$$

When $0 < r < 1$, the image is the negative v—axis, and when $r > 1$, the image is the positive v-axis, again crossing the branch cut and hence changing sheets. The same phenomenon repeats at $\theta = 3\pi/2$.

We can now summarize the above observations. We take two sheets of the uv-plane. Call them sheet A and sheet B. Cut both along the real interval $[-2, 2]$. Glue the upper part of the cut on sheet A to the lower part of the cut on sheet B, and glue the lower part of the cut on sheet A to the upper part of the cut on sheet B. The points ± 2 are the branch points.

The interval $(0, 1]$ goes to the interval $[2, \infty)$ on sheet A in reverse order. The interval $[>, \infty)$ goes to interval $[2, \infty)$ on sheet B. A circle with radius $0 < r < 1$ goes to an ellipse in sheet A with major axis $r + 1/r > 2$, traveling clockwise around the origin. The upper part of the unit circle maps to the interval $[-2, 2]$ which is the lower part of the cut on sheet A and also the upper part of the cut on sheet B. The interval $[-2, 2]$ is traced from 2 to -2 as the unit circle travels counterclockwise from $\theta = 0$ to $\theta = \pi$. A circle of radius $r > 1$ around the origin in z-plane is mapped to an ellipse as above but this time in the B sheet. A ray with $0 < \theta < \pi/2$ is mapped to a hyperbola the lower arm of which lies in the fourth quadrant of the w-plane and intersects the lower part of the cut at $2\cos\theta$ and emerging from the upper part of the cut at the same point in sheet B. The upper part of the hyperbola then continues in sheet B. See the figure at next page.



Sheet A
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-2
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v
Sheet B
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-2
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