Due Date: 1 December 2014, Monday – Class time NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Homework 4 – Solutions

1	2	3	4	5	TOTAL
100	0	0	0	0	100

Please do not write anything inside the above boxes!

Check that there is **1** question on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) [Conway, page 133, Exercise 5] Let f be analytic in $D = \{z \mid |z| < 1\}$ and suppose that $|f(z)| \leq M$ for all z in D.

(a) If $f(z_k) = 0$ for $1 \leq k \leq n$ show that

$$|f(z)| \leq M \prod_{k=1}^{n} \frac{|z - z_k|}{|1 - \bar{z}_k z|}$$

for $|z| < 1$.

(b) If $f(z_k) = 0$ for $1 \leq k \leq n$, each $z_k \neq 0$, and $f(0) = M(z_1 z_2 \cdots z_n)$, find a formula for f.

Solution:

Set

$$\phi(z) = \phi_{z_1}(z) \cdots \phi_{z_n}(z),$$

where $\phi_a(z) = \dfrac{z - a}{1 - \bar{a}z}$. It follows that ϕ is analytic on D and that $\phi(D) \subseteq D$. Then

$$h(z) = \frac{f(z)}{\phi(z)}$$

is analytic on D. We claim that $|h(z)| \leq M$ for all $z \in D$. Assume not. Then there exists a point $z_0 \in D$ such that

$$|h(z_0)| = N > M.$$

Now for any r with $|z_0| < r < 1$ define

$$\nu_r = \min_{|z|=r} |\phi(z)|.$$

Since $|\phi(z)| = 1$ for all $z \in \partial D$, we have by the maximum modulus principle that

$$\lim_{r \to 1^-} \nu_r = 1.$$

Again by the maximum modulus principle we have, for $|z_0| < r < 1$,

$$\max_{|z|=r} |h(z)| \geq N = |h(z_0)|.$$

Combining these we have, for any z with $|z| = r$ with $|z_0| < r < 1$,

$$M \geq |f(z)| = |\phi(z)|\,|h(z)| \geq \nu_r\,|h(z)|.$$

This gives

$$M \geq \nu_r\,|h(z)| \quad \text{for } |z| = r.$$

Taking the maximum of both sides for $|z| = r$ we have

$$M \geq \nu_r \max_{|z|=r} |h(z)| \geq \nu_r N.$$

This gives

$$M \geq \nu_r \, N,$$

and taking the limit of both sides as r goes to 1, we get

$$M \geq N,$$

which contradicts the assumption $N > M$ which we had above. This proves that

$$|h(z)| \leq M \quad \text{for all } z \in D.$$

The rest is now trivial. From $f(z) = \phi(z)h(z)$ we get

$$|f(z)| \leq M \prod_{k=1}^{n} \frac{|z - z_k|}{|1 - \bar{z}_k z|}$$

for $|z| < 1$, as required in part (a).

For part (b) note that

$$\phi(0) = (-1)^n (z_1 \cdots z_n) \quad \text{and hence} \quad h(0) = (-1)^n M,$$

following the assumption about $f(0)$. But this gives $|h(0)| = M$, which says that the maximum value of the modulus is attained at an interior point. Then by the maximum modulus principle, h must be constant.

$$h(z) = h(0) = (-1)^n M.$$

Since $h = f/\phi$, we have

$$f(z) = (-1)^n M \phi(z) = (-1)^n M \prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z}_k z}.$$