Due Date: 29 December 2014, Monday – Class time NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 503 Complex Analysis – Homework 5 – Solutions

1	2	3	4	5	TOTAL
100	0	0	0	0	100

Please do not write anything inside the above boxes!

Check that there is **1** question on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Prove the following formula for $\operatorname{Re} z > 0$.

$$\Gamma(z)\,\frac{\sin\theta z}{n\,(a^2+b^2)^{z/2}} \quad = \quad \int_0^\infty e^{-at^n} t^{nz-1}\sin(bt^n)\,dt$$

where n is a positive integer, a and b are real numbers with $(a,b)\neq(0,0)$, and $\tan\theta = b/a$. When $a=0$, we take $\theta=\pm\pi/2$ such that $\theta b>0$.
[Hint: Start with $\Gamma(z)=\int_0^\infty e^{-s}s^{z-1}ds$ and make the substitution $s=(a+ib)t^n$.]

Using this formula evaluate

$$\int_0^\infty \sin t^n\,dt.$$

Solution:

Letting $s=(a+ib)t^n$ we get $ds = n(a+ib)t^{n-1}dt$. This gives

$$\Gamma(z)=\int_0^\infty e^{-s}s^{z-1}ds = n(a+ib)^z\int_0^\infty e^{-at^n}t^{nz-1}e^{-ibt^n}\,dt.$$

Using Euler formula

$$e^{-ibt^n} = \cos(bt^n) - i\sin(bt^n),$$

we obtain

$$\frac{\Gamma(z)}{n(a+ib)^z} = \int_0^\infty e^{-at^n}t^{nz-1}\cos(bt^n)dt - i\int_0^\infty e^{-at^n}t^{nz-1}\sin(bt^n)dt. \tag{A}$$

Similarly, starting with the substitution $s=(a-ib)t^n$, we get

$$\frac{\Gamma(z)}{n(a-ib)^z} = \int_0^\infty e^{-at^n}t^{nz-1}\cos(bt^n)dt + i\int_0^\infty e^{-at^n}t^{nz-1}\sin(bt^n)dt. \tag{B}$$

Let $(a+ib)=re^{i\theta}$ where $r=\sqrt{a^2+b^2}$ and θ is as above. Then we have

$$\frac{1}{(a+ib)^z} = \frac{e^{-i\theta z}}{r^z} = \frac{\cos(\theta z)}{r^z} - i\frac{\sin(\theta z)}{r^z}.$$

Similarly

$$\frac{1}{(a-ib)^z} = \frac{e^{i\theta z}}{r^z} = \frac{\cos(\theta z)}{r^z} + i\frac{\sin(\theta z)}{r^z}.$$

Finally, adding both sides of equations (A) and (B), we get

$$\Gamma(z)\,\frac{\cos\theta z}{n\,(a^2+b^2)^{z/2}} = \int_0^\infty e^{-at^n}t^{nz-1}\cos(bt^n)\,dt.$$

And subtracting equation (A) from equation (B) we get

$$\Gamma(z)\,\frac{\sin\theta z}{n\,(a^2+b^2)^{z/2}} = \int_0^\infty e^{-at^n}t^{nz-1}\sin(bt^n)\,dt.$$

In this last equation set $a=0$ and $b=1$. Then $\theta=\pi/2$. Also set $z=1/n$. We get

$$\Gamma(1/n)\frac{\sin(\pi/(2n))}{n} = \int_0^\infty \sin(t^n)\,dt.$$