Due Date: 17 November 2014, Monday – Class time NAME:..

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Math 503 Complex Analysis – Take-Home Midterm Exam 1 – Solutions

1	2	3	4	TOTAL
20	20	20	40	100

Please do not write anything inside the above boxes!

Check that there are **2** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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Q-1) For a fixed integer $n > 0$ and a fixed real number $\alpha > 0$, find all entire functions f satisfying $|f(z)| \leq \alpha |z|^n$ for all $z \in \mathbb{C}$.

Solution:

Let $h(z) = f(z)/z^n$. This is an analytic function on the plane with a singularity at $z = 0$. The fact that $|h(z)| \leq \alpha$, for all $z \in \mathbb{C}$, shows that
(a) $z = 0$ is a removable singularity for h, hence h is entire and
(b) by Liouville's theorem h is constant.
Let this constant be $\alpha' \in \mathbb{C}$. Then $f(z) = \alpha' z^n$ and since $|f(z)| \leq \alpha |z|^n$, we must have $|\alpha'| \leq \alpha$.

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Q-2) Note that $\operatorname{cotan} z$ is a meromorphic function with a simple pole at each $z = \pi n$, where $n \in \mathbb{Z}$. Therefore its Laurent series

$$\operatorname{cotan} z = \frac{b_1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

converges for $|z| < \pi$. Determine the coefficients $b_1, a_0, a_1, \ldots, a_n, \ldots$.
The standard and easiest way to do this to use the following facts:
(a) $e^{iz} = \cos z + i \sin z$, for all $z \in \mathbb{C}$, and
(b) $\dfrac{z}{e^z - 1} = \displaystyle\sum_{n=0}^{\infty} \frac{B_n}{n!}\, z^n$, for $|z| < 2\pi$, where B_n are Bernoulli numbers with the convention that $B_0 = 1$ and $B_1 = -\dfrac{1}{2}$.

Solution:

First observe that using *(a)* above we can write

$$\operatorname{cotan} z = \frac{\cos z}{\sin z} = i\,\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = i\,\frac{e^{2iz} + 1}{e^{2iz} - 1} = i\,\left(1 + \frac{2}{e^{2iz} - 1}\right) = i\,\left(1 + \frac{1}{iz}\frac{2iz}{e^{2iz} - 1}\right).$$

Next, using *(b)* above we can further write

$$\operatorname{cotan} z == i\,\left(1 + \frac{1}{iz}\frac{2iz}{e^{2iz} - 1}\right) == i\,\left(1 + \frac{1}{iz}\sum_{m=0}^{\infty} \frac{B_m}{m!}\,(2iz)^m,\right)$$

which now converges for $|z| < \pi$. Simplifying this, and noting that $B_{2n+1} = 0$ for $n \geq 1$, we get finally

$$\operatorname{cotan} z = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{4^n B_{2n}}{(2n)!}\, z^{2n-1} = \frac{1}{z} - \frac{1}{3}\,z - \frac{1}{45}\,z^3 - \frac{2}{945}\,z^5 - \frac{1}{4725}\,z^7 - \frac{2}{93555}\,z^9 - \cdots.$$

Note that all the upcoming signs are negative since $B_{2n} = (-1)^{n-1} |B_{2n}|$ for $n \geq 1$.

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Q-3) Let U be a non-empty, open and connected subset of $\mathbb{C}$, and let f be a holomorphic map on U. Assume that there is a point $z_0 \in U$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in U$.

1. Using Cauchy Integral Formula, show that $|f(z)| = c$, a constant, for all $z \in U$.
2. Using Cauchy-Riemann equations, show that f is constant, assuming that $|f(z)|$ is constant.
3. Using the Open Mapping Theorem, show that f is constant, assuming that $|f(z)|$ is constant.

Solution:

Let $r > 0$ be such that $B_r(z_0) \subset U$. The Cauchy Integral Formula gives us

$$f(z_0) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

Taking absolute values of both sides, we get

$$|f(z_0)| \leq |f(z_0 + re^{i\theta})| \leq |f(z_0)|,$$

where the last inequality is the given fact about $f(z_0)$. But this forces $|f(z)|$ to be constant throughout $B_r(z_0)$. Now let z_1 be any other point of U and let γ be a path connecting z_0 to z_1. We repeat the above argument for every point on γ and conclude that $|f(z_0)| = |f(z_1)|$. This shows that $|f(z)| = c$, constant, for all $z \in U$.

If $c = 0$, then clearly $f = 0$ and is constant, so there is nothing to prove. Assume $c \neq 0$.

Let $f(z) = u(x, y) + iv(x, y)$, where u and v are real functions. We found that

$$u^2 + v^2 = c^2.$$

Taking partial derivatives of both sides with respect to x and y separately, we find

$$u\, u_x + v\, v_x = 0 \text{ and } u\, u_y + v\, v_y = 0.$$

Using Cauchy-Riemann equations to replace v_x and v_y with $-u_y$ and u_x respectively, we get

$$u\, u_x - v\, u_y = 0 \quad \text{and} \quad u\, u_y + v\, u_x = 0. \tag{$*$}$$

Multiply the first equation by u, the second by v, and add side by side to obtain

$$(u^2 + v^2)\, u_x = 0,$$

which implies that $u_x = 0$ since $c \neq 0$. Next again starting from equation ($*$), multiply the first equation by v, the second by u and subtract to obtain

$$(u^2 + v^2)\, u_y = 0,$$

which implies that $u_y = 0$. Hence we get $u = k$, a constant. Now Cauchy-Riemann equations give that $v_x = v_y = 0$, so $v = k'$ is also constant. This finally shows that $f(z) = k + ik'$ is constant.

If we however use a strong theorem such as the Open Mapping Theorem, we can prove that f is constant much more easily. Since $|f(z)| = c$, the image of U under f is the closed set $u^2 + v^2 = c^2$. This is possible only if f is constant.

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Q-4) For any $\alpha \in \mathbb{R}$, define the integral

$$I_\alpha = \int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}}\, dx.$$

Show that I_α exists if and only if $0 < \alpha < 2$, and in that case we have

$$I_\alpha = \frac{\pi}{\alpha}\ \operatorname{cosec}(\frac{\pi}{2}\ \alpha).$$

Solution:

The singularities of the integrand are essential singularities. We try integration by parts with the hope of getting a better integrand.

$$u = \log(1+x^2) \qquad dv = \frac{dx}{x^{1+\alpha}}$$

$$du = \frac{2x\, dx}{1+x^2} \qquad v = -\frac{1}{\alpha x^\alpha},$$

so we get

$$\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}}\, dx = -\frac{1}{\alpha}\left(\frac{\log(1+x^2)}{x^\alpha}\bigg|_0^\infty \right) + \frac{2}{\alpha}\int_0^\infty \frac{x^{1-\alpha}}{1+x^2}\, dx.$$

Therefore the existence of I_α first of all requires the existence of the limits $\left(\frac{\log(1+x^2)}{x^\alpha}\Big|_0^\infty \right)$. By using L'Hospital's rule we find that

$$\left(\frac{\log(1+x^2)}{x^\alpha}\bigg|_0^\infty \right) = \begin{cases} 0 & \text{if } 0<\alpha<2, \\ -1 & \text{if } \alpha = 2, \\ \infty & \text{otherwise.} \end{cases} .$$

When $\alpha = 2$, we see that

$$\int \frac{1}{x(1+x^2)}\, dx = \int \left(\frac{1}{x} - \frac{x}{1+x^2} \right)\, dx = \frac{1}{2}\ln\left(\frac{x^2}{1+x^2} \right) + C.$$

Calculating the limits, we see that

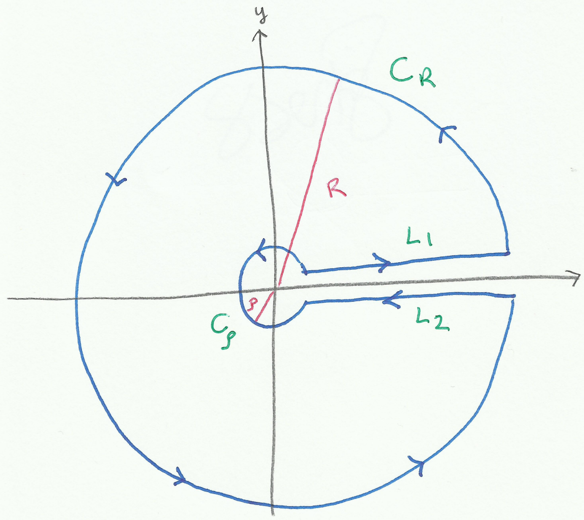
$$\lim_{x\to\infty} \ln\left(\frac{x^2}{1+x^2} \right) = 0, \quad \text{but} \quad \lim_{x\to 0^+} \ln\left(\frac{x^2}{1+x^2} \right) = -\infty.$$

Therefore I_2 does not exist.

We now assume $0<\alpha<2$, and check if I_α exists for these values of α. Note that for these values of α, we have

$$I_\alpha = \frac{2}{\alpha}\int_0^\infty \frac{x^{1-\alpha}}{1+x^2}\, dx. \qquad (**)$$

To evaluate this integral first consider the function $f(z) = \dfrac{z^{1-\alpha}}{1+z^2}$, and the following contour.



$\gamma_{\rho,R} = L_1 + C_R + L_2 + C_\rho$

we use that branch of log where $z = re^{i\theta}$ with $0 \leq \theta \leq 2\pi$, and $\log z = \ln r + i\theta$.

Here $0 < \rho < 1 < R$, and C_ρ, C_R are circles centered at the origin with radii ρ and R, respectively. L_1 is the line parametrized by $z = x$, with $x \in (\rho, R)$, and $-L_2$ is the line parametrized by $z = xe^{2\pi i}$, with $x \in (\rho, R)$. The $e^{2\pi i}$ factor in L_2 comes from the fact that we turn around the origin once and the logarithm function keeps a record of this. Integer powers of z will not notice this turn but $z^{1-\alpha}$ will certainly incorporate non-trivially the turning factor $e^{2\pi i}$, when $\alpha \neq 1$, as we will see below.

Let $\gamma_{\rho,R} = L_1 + C_R + L_2 + C_\rho$ be the indented path in the figure. We will calculate $\displaystyle\int_{\gamma_{\rho,R}} f(z)dz$ in two different ways and equate them with the hope of recovering I_α somewhere during the process. First we observe that

$$\int_{\gamma_{\rho,R}} f(z)dz = \int_{L_1} f(z)dz + \int_{C_R} f(z)dz + \int_{L_2} f(z)dz + \int_{C_\rho} f(z)dz.$$

When we take the limit of both sides as $\rho \to 0$ and $R \to \infty$ we will recover I_α on the right hand side, and it will be equated to the other calculation of $\displaystyle\int_{\gamma_{\rho,R}} f(z)dz$ as done below.

$$\int_{\gamma_{\rho,R}} f(z)dz = (2\pi i)(\quad \text{sum of residues of } f(z) \text{ inside the loop} \quad).$$

Now we start to calculate the integrals on the circles. For this purpose let $K > 0$ be any real number with $K \neq 1$, and let C_K be the circle centered at the origin with radius K. The modulus of the integral of f around C_K can be bounded as follows.

$$\left|\int_{C_K} f(z)\,dz\right| = \left|\int_{|z|=K} \frac{z^{1-\alpha}}{1+z^2}\,dz\right| \leq 2\pi K\,\frac{K^{1-\alpha}}{|1-K^2|} = 2\pi\,\frac{K^{2-\alpha}}{|1-K^2|}.$$

At this point observe that

$$\lim_{K\to 0}\frac{K^{2-\alpha}}{|1-K^2|}=0 \quad \text{since} \quad 2-\alpha>0,$$

and

$$\lim_{K\to \infty}\frac{K^{2-\alpha}}{|1-K^2|}=0 \quad \text{since} \quad 2-\alpha<2.$$

Therefore we have

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz=0, \quad \text{and} \quad \lim_{R\to\infty}\int_{C_R} f(z)dz=0.$$

We now calculate the integrals on L_1 and L_2. On L_1, we have $z=x$ and $x\in[\rho,R]$, so

$$\int_{L_1} f(z)\,dz=\int_\rho^R \frac{x^{1-\alpha}}{1+x^2}\,dx.$$

On $-L_2$, we have $z=xe^{2\pi i}$, with $x\in[\rho,R]$, so we have

$$f(z)\,dz=\frac{(xe^{2\pi i})^{1-\alpha}}{1+(xe^{2\pi i})^2}\,d(xe^{2\pi i})=e^{-2\pi i\alpha}\,\frac{x^{1-\alpha}}{1+x^2}\,dx.$$

Hence we have

$$\int_{L_2} f(z)\,dz=-\int_{L_2} f(z)\,dz=-e^{-2\pi i\alpha}\int_\rho^R \frac{x^{1-\alpha}}{1+x^2}\,dx.$$

This leads to the equality

$$\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\gamma_{\rho,R}} f(z)\,dz=\frac{\alpha}{2}\left(1-e^{-2\pi i\alpha}\right)I_\alpha, \qquad (***)$$

where we used equation ($**$).

Now we calculate residues. We use the branch of the $\log$ function compatible with our region inside the loop $\gamma_{\rho,R}$. This means that we write z in polar coordinates as $z=re^{i\theta}$ with $\theta\in[0,2\pi]$. In particular we have

$$\log i=\frac{\pi}{2}\,i, \quad \text{and} \quad \log(-i)=\frac{3\pi}{2}\,i.$$

We then have

$$\begin{aligned}
\operatorname{Res}\left(\frac{z^{1-\alpha}}{1+x^2},z=i\right) &= \left.\frac{z^{1-\alpha}}{2z}\right|_{z=i}=\left.\frac{z^{-\alpha}}{2}\right|_{z=i}=\frac{1}{2}\,i^{-\alpha}=\frac{1}{2}e^{-\alpha\log i}=\frac{1}{2}e^{-\alpha\frac{\pi}{2}\,i}\\
&= \frac{1}{2}\cos\frac{\pi}{2}\,\alpha-i\frac{1}{2}\sin\frac{\pi}{2}\,\alpha,\\
\operatorname{Res}\left(\frac{z^{1-\alpha}}{1+x^2},z=-i\right) &= \left.\frac{z^{1-\alpha}}{2z}\right|_{z=-i}=\left.\frac{z^{-\alpha}}{2}\right|_{z=-i}=\frac{1}{2}\,(-i)^{-\alpha}=\frac{1}{2}e^{-\alpha\log(-i)}=\frac{1}{2}e^{-\alpha\frac{3\pi}{2}\,i}\\
&= \frac{1}{2}\cos\frac{3\pi}{2}\,\alpha-i\frac{1}{2}\sin\frac{3\pi}{2}\,\alpha.
\end{aligned}$$

Therefore

$$(2\pi i)(\text{sum of residues})=\pi(\sin\frac{\pi}{2}\,\alpha+\sin\frac{3\pi}{2}\,\alpha)+i\pi(\cos\frac{\pi}{2}\,\alpha+\cos\frac{3\pi}{2}\,\alpha).$$

This gives

$$\int_{\gamma_{\rho,R}} f(z)\,dz = \pi(\sin\frac{\pi}{2}\,\alpha + \sin\frac{3\pi}{2}\,\alpha) + i\pi(\cos\frac{\pi}{2}\,\alpha + \cos\frac{3\pi}{2}\,\alpha).$$

Note that the right hand side is independent of ρ and r, so taking limits of both sides as $\rho \to 0$ and $R \to \infty$, and using equation $(***)$, we get

$$\frac{\alpha}{2}\left(1 - e^{-2\pi i\alpha}\right) I_\alpha = \pi(\sin\frac{\pi}{2}\,\alpha + \sin\frac{3\pi}{2}\,\alpha) + i\pi(\cos\frac{\pi}{2}\,\alpha + \cos\frac{3\pi}{2}\,\alpha).$$

Since $1 - e^{-2\pi i\alpha} = (1 - \cos 2\pi\alpha) + i(\sin 2\pi\alpha)$, we have

$$\frac{\alpha}{2}(1 - \cos 2\pi\alpha)I_\alpha = \pi(\sin\frac{\pi}{2}\,\alpha + \sin\frac{3\pi}{2}\,\alpha) \tag{A}$$

and

$$\frac{\alpha}{2}(\sin 2\pi\alpha)I_\alpha = \pi(\cos\frac{\pi}{2}\,\alpha + \cos\frac{3\pi}{2}\,\alpha). \tag{B}$$

When $\alpha \neq \frac{1}{2}, 1, \frac{3}{2}$, then we can divide by $\sin 2\pi\alpha$ in equation (B) and get

$$\begin{aligned} I_\alpha &= \frac{2\pi}{\alpha}\frac{\cos\frac{\pi}{2}\,\alpha + \cos\frac{3\pi}{2}\,\alpha}{\sin 2\pi\alpha} \\ &= \frac{2\pi}{\alpha}\frac{\cos\pi\alpha\ \cos\frac{\pi}{2}\alpha}{\sin\pi\alpha\ \cos\pi\alpha} \\ &= \frac{2\pi}{\alpha}\frac{\cos\frac{\pi}{2}\alpha}{\sin\pi\alpha} \\ &= \frac{2\pi}{\alpha}\frac{\cos\frac{\pi}{2}\alpha}{2\sin\frac{\pi}{2}\alpha\ \cos\frac{\pi}{2}\alpha} \\ &= \frac{\pi}{\alpha}\,\mathrm{cosec}\,\frac{\pi}{2}\alpha. \end{aligned}$$

Notice that the above cancelations were possible since $\alpha \neq \frac{1}{2}, 1, \frac{3}{2}$.

Now suppose $\alpha = \frac{1}{2}$ or $\alpha = \frac{3}{2}$. Equation (A) gives immediately that

$$I_\alpha = \frac{\pi}{\alpha}\,\sqrt{2}.$$

Note however that

$$\mathrm{cosec}\,\frac{\pi}{2}\alpha = \sqrt{2} \quad \text{when } \alpha = \frac{1}{2} \quad \text{or} \quad \frac{3}{2}.$$

When $\alpha = 1$, we go back to equation $(**)$, to obtain

$$I_1 = 2\int_0^\infty \frac{x}{1+x^2}\,dx = 2\left(\arctan x\Big|_0^\infty\right) = \pi.$$

Again notice that

$$\mathrm{cosec}\,\frac{\pi}{2}\alpha = 1 \quad \text{when } \alpha = 1 \quad .$$

Hence our final formula is

$$I_\alpha = \frac{\pi}{\alpha}\,\mathrm{cosec}\,\frac{\pi}{2}\alpha, \quad \text{for} \quad 0 < \alpha < 2.$$